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Y3/Y4 Formation and Evolution of Galaxies

Revision Problem Sheet 1
(with solutions)

1. In a spherical galaxy, the circular velocity is given by

$$v_c^2 = \frac{a\,r^2}{(r^2+b^2)^{3/2}},$$

where a and b are constants.

(a) Find an expression for the gravitational potential of this galaxy as a function of radius r. [3]

$$\frac{d\Phi}{dr} = -\frac{v_c^2}{r} = -\frac{a\,r}{(r^2+b^2)^{3/2}}.$$

Integrating from $r = 0$ to r,

$$\Phi(r) = -\frac{a}{(r^2+b^2)^{1/2}}.$$

(b) Show that for large radii $r \gg b$, the potential approaches that of a point mass. [2]

For radii $r \gg b$, the above reduces to $\Phi(r) \propto -\frac{a}{r}$.

(c) Using Poisson's equation, find the density of the galaxy as a function of the radius r. [2]

Use Poisson's Equation in spherical coordinates. Since the problem is spherically symmetric, only the d/dr terms are relevant.

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G\rho$$

Use the expression of v_c^2 given, noticing that $r(d\Phi/dr) = v_c^2$,

$$r^2\frac{d\Phi}{dr} = \frac{a\,r^3}{(r^2+b^2)^{3/2}}.$$

$$\rho\left(r\right)=\frac{a}{4\pi G}\frac{3b^{2}}{\left(r^{2}+b^{2}\right)^{5/2}},$$

(d) Using the substitution $r = b\tan\theta$, show that the mass of this galaxy is finite, and find its value. [3]

$$\text{Total Mass} = \int_0^\infty 4\pi r^2\rho dr = \frac{3ab^2}{4\pi G}.4\pi\int_0^\infty \frac{r^2dr}{\left(r^2+b^2\right)^{5/2}}.$$

With $r = b\tan\theta$, one gets $r^2 + b^2 = b^2\sec^2\theta$, so the integral turns into

$$\frac{3a}{G}\int_0^{\pi/2}\sin^2\theta\cos\theta d\theta = \frac{a}{G}.$$

2. The relaxation time for the cluster may be expressed as

$$t_{\rm relax}\sim\frac{1}{8N\ln N}\frac{v^3D^3}{(Gm)^2},$$

where D is the size of the cluster, N the number of stars, m their typical mass, v their typical speed, and G is the Gravitational constant.

(a) Show that the ratio between relaxation time and crossing time is approximately $N/8\ln N$. [4]

For a cluster of N stars of mass m mean velocity v and size D, the crossing time is $t_{\rm cross} = D/v$.

From the Virial theorem, $2T + V = 0$, where $T = Nmv^2/2$ and $V = G(Nm)^2/D$, so $v^2 = NGm/D$. Result (a) follows.

(b) A cluster of galaxies of radius 0.5 Mpc consists of 300 galaxies and has a one-dimensional velocity dispersion of 1000 km s^{-1}. Estimate both its crossing time and relaxation time. Compare these estimates to the age of the Universe and comment on the expected dynamical state of the cluster. [6]

A one-dimensional velocity dispersion $\sigma_r = 1000$ km/s corresponds to a spatial velocity dispersion of $\sqrt{3}\times 1000 = 1732$ km/s. The crossing time is $t_{\rm cross} = D/v = 1$ Mpc/1732 km/s $\approx 5.6\times 10^9$ yr.

The two-body relaxation time $t_{\rm relax} = N/8\ln N\times t_{\rm cross}$. Here $N = 300$, so $t_{\rm relax} = 3.7\times 10^{10}$ yr.

3. Find the speed v_c of a star moving in a circular orbit about the centre of a singular isothermal sphere, of velocity dispersion σ (independent of radius), where the radial distribution of mass is given by $M(r) = 2\sigma^2 r/G$. [10]

The gravitational force per unit mass is related to the circular speed by

$$|f(r)| \equiv \frac{v_c^2}{r},$$

which means

$$\frac{v_c^2}{r} = \frac{GM(r)}{r^2} = \frac{2\sigma^2}{r}.$$

Thus $v_c = \sqrt{2}\sigma$.

4. A gas cloud of density ρ and sound speed c_s is collapsing under its own gravity. If the dispersion relation for acoustic waves is $\omega^2 = k^2 c_s^2 - 4\pi G\rho$, where the $k \equiv 2\pi/\lambda$ is the wavenumber, derive the Jeans length of the cloud in terms of the given quantities. What is the physical significance of the Jeans length in the context of the evolution of the cloud? [10]

The dispersion relation would be $\omega^2 = k^2 c_s^2 - 4\pi G\rho$, where k is the wavenumber $\equiv 2\pi/\lambda$. Negative w^2 implies growing perturbations, so Jeans instability sets in when $4\pi G\rho > k^2 c_s^2$. The Jeans length for the gas cloud thus is

$$\lambda_J = c_s \left(\frac{\pi}{G\rho}\right)^{1/2}.$$

Perturbations of length scale larger than λ_J are unstable, and so will collapse to form distinct objects. The energy density of an ordinary sound wave is positive, However, the gravitational energy density of a sound wave is negative, since the enhanced attraction in the compressed regions overwhelms the reduced attraction in the dilated regions. The Jeans instability sets in where the net energy density becomes negative, so that the system can evolve to a lower energy state by allowing the wave to grow, and thus the system to fragment.

5. Outline the observational evidence in favour of the presence of a supermassive black hole at the centre of the Milky way galaxy. Consider in particular the evidence from direct observations, the dynamics of the immediate surroundings of the Galactic centre, and high-energy observations resulting from the accretion properties of the candidate black hole. [10]

Read your text books!