Formation and Evolution of galaxies

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An sketchy and far-from-complete set of notes for
Y3/Y4 students (2011-12)

Useful Conversions:

1 erg	10^{-7} Joules
1 parsec (pc)	3.086×10^{16} m
1 radian	$206,265$ arcsec
1 year	3.16×10^{7} s
	$\simeq \pi \times 10^{7}$ s
Solar Mass $M_\odot$	1.99×10^{30} kg
Solar Radius $R_\odot$	6.96×10^{8} m
Solar Luminosity $L_\odot$	3.90×10^{26} J s^{-1}
Solar Absolute Magnitude M_v	4.83
Gravitaional constant G	$4.98 \times 10^{-15}\ M_\odot^{-1}\,\mathrm{pc}^3\,\mathrm{yr}^{-2}$

Useful Definitions:

Apparent Magnitude m from flux f	$m_1 - m_2 = -2.5\ \log_{10} \frac{f_1}{f_2}$
Absolute magnitude M from distance D	$m - M = 5\log_{10} D_{\mathrm{Mpc}} + 25$

Convenient Units:

	Galaxies	Clusters of Galaxies
G	1	1
Length L	1 kpc	1 Mpc
Speed v	1 km/s	100 km/s
Time t	(1 kpc)/(1 km/s) $= 0.97 \times 10^{9}$ yr	(1 Mpc)/(100 km/s) $= 0.97 \times 10^{10}$ yr
Mass M	(1/G) $\times$ 1 kpc $\times$ (km/s)^2 $= 2.32 \times 10^{5}\ M_\odot$	(1/G) $\times$ 1 Mpc $\times$ (100 km/s)^2 $= 2.32 \times 10^{12}\ M_\odot$

1. Galaxies

Galaxies are the basic building blocks of matter in the Universe, though this wasn't realized till the 1930s. Compared to our understanding of the physics of stars, galaxies are still poorly understood, though our knowledge of galaxies is improving significantly as we speak[1].

Galaxies are difficult to understand not only because they are made of three very different constituents, but also that they are much more than the sum of their parts. There are stars, the physics of which is relatively better understood, but there's also the interstellar medium (gas and dust, which produces stars, and is in turn fed by dying stars), and dark matter (about which we know very little, except that it's there). These three very different kinds of things all interact with each other, and interaction with other galaxies and the local extragactic environment crucially affects the evolution of galaxies. Some galaxies (more of them in earlier epochs) have 'active nuclei' which can vastly outshine the starlight, but here we will mostly deal with normal galaxies[2].

Dynamical timescales

Consider an isolated cluster of N stars each of mass m, where the average speed of the stars with respect to the centroid of the cluster is v. One can assume that the average separation between these stars is roughly half the size D of the system.

The virial theorem (we'll prove this later) states that for any system bound by an inverse square force, the time-averaged kinetic energy $\langle T\rangle$ and the time-averaged potential energy $\langle V\rangle$ satisfy $2\langle T\rangle + \langle V\rangle = 0$. For the system in question, this implies that $2 \times \frac{1}{2}Nmv^2 \approx G(Nm)^2/D$, which means

$$v^2 \simeq GNm/D. \tag{1.1}$$

1. **Crossing time:** The time taken by a star, moving with the average speed, to cross from one side of the cluster to the other provides an useful timescale.

$$t_{\rm cross} = \frac{D}{v} \approx \left(\frac{D^3}{GNm}\right)^{\frac{1}{2}}.$$

   This is also known as the *dynamical timescale* of the system.

Example [Crossing times for systems of stars and galaxies] For Globular clusters, assuming $N = 10^6$, $D = 20$ pc and a mean mass for its stars $m = 0.5\ M_\odot$, the crossing time turns out to be $t_{\rm cross} = 2 \times 10^6$ yr. However, for the core of a cluster of galaxies, where $v = 1000$ km s^{-1}, $D = 2$ Mpc, the crossing time would be $t_{\rm cross} \equiv D/v = 2 \times 10^{10}$ yr. Thus globular clusters of stars are well mixed, whereas clusters of galaxies are necessarily not so. □

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[1] The standard texts/references are *Galaxies in the Universe* by Sparke and Gallagher, *Galactic Dynamics* by Binney and Tremaine, and *Galactic Astronomy* by Binney and Merrifield. *The Physical Universe* by Shu is more elementary, but very insightful and always repays reading. For those of you who haven't taken an astronomy module before, Chapters 12-13 of Shu are essential reading.

[2] Those of you who are 4th-year M.Sci. Students will deal with some aspects of active galaxies.

2. **Relaxation time:** The two-body relaxation time $t_{\rm relax}$ is the time taken for a star's velocity to be changed significantly changed from two-body interactions, such that $\Delta v^2 \simeq v^2$. Most textbooks would have a derivation- it turns out

$$T_{\rm relax} == \frac{1}{8N \ln N} \frac{(Dv)^3}{(Gm)^2}, \tag{1.2}$$

where N is the total number of stars in the system of size D, and m and v the typical mass and speed of a star.

A BACK-OF-THE-ENVELOPE CALCULATION: The v and m dependences of the relaxation time can be found in a order-of-magnitude calculation. Consider N stars of mass m each in a box of side R, and let these stars be fixed. Then send another star through this box with speed v. The moving star needs to pass within $r_0 \simeq Gm/v^2$ of one of the fixed stars, so that kinetic and two-body potential energies are equal. In time t the expectation value of the number of stars passing within distance r_0 is $\pi r_0^2 vt$. Equating this expectation value to unity gives the time, of order

$$\frac{(Rv)^3}{N(Gm)^2}.$$

The $\ln N$ term comes from the fact that relaxation is a cumulative effect from two-body encounters with all stars. More distant encounters each have less effect, but there are more of them. So these more distant encounters shortens the relaxation time by a factor depending weakly on N.

It's easier to remember $T_{\rm relax}$ in crossing times. Taking $R/b_{\rm min} \simeq N$ and then using equation (1.1) to eliminate R, we get

$$\frac{T_{\rm relax}}{T_{\rm cross}} \simeq \frac{N}{8 \ln N}. \tag{1.3}$$

Galaxies are $\lesssim 10^3 T_{\rm cross}$ old and have $\gtrsim 10^6$ stars, so stellar encounters have negligible dynamical effect. In globular clusters, which may have $\sim 10^6$ stars and be $\sim 10^5$ crossing times old, stellar encounters start to become important, and in the cores of globular clusters two-body relaxation is very important.

Stellar collisions: Another way of looking at this is to imagine that each star has a "sphere of influence" of radius r_s around its centre. We would call an event an "encounter" if the centres of two stars came closer to each other than r_s. We could define ***strong*** encounters to occur when two stars come close enough such that their mutual potential energy is of the same order as their kinetic energy, *i.e.* $\frac{1}{2}mv^2 \approx Gm^2/r_s$, which implies that their separation

$$r \lesssim r_s = \frac{2Gm}{v^2}.$$

For the solar neighbourhood, where $v = 30$ km/s, this quantity is about 1 AU.

If the local density of stars is n, and a typical random speed v, then the cylindrical volume swept out by a single star's sphere of influence is $\pi r^2 v t_s$. This volume can contain just one other star (if it had more, then in the same time interval there would have been more collisions). Therefore, $n(\pi r^2 v t_s) = 1$, or,

$$t_s = \frac{1}{n\pi r_s^2 v} = \frac{v^3}{4\pi G^2 m^2 n} = \frac{v^3 D^3}{3\, G^2 m^2 N},$$

for a spherical stellar system of size D, where $N = \frac{4}{3}\pi R^3 n$. For the solar neighbourhood, where $v = 30$ km/s, $n = 0.1\,\mathrm{pc}^{-3}$ and $m = 0.5\,M_\odot$, $t_s \sim 5 \times 10^{15}$ yr. To calculate the freqeuncy of actual "collisions", substitute r_s by the radius of a star (about 7×10^8 m for the Sun), and one can see why collisions between stars are so infrequent ($t_{\mathrm{collision}} \sim 10^{19}$ yr).

From this we can conclude that stars are so compact on the scale of a galaxy that a stellar system behaves like a collisionless fluid (except in the cores of galaxies and globular clusters), resembling a plasma in some respects. Gas and dust are collisional. This leads to two very important differences between stellar and gas dynamics in a galaxy.

1) Gas tends to settle into disks, but stars don't.
2) Gravity must be balanced by motion in stellar and gas dynamics, but in equilibrium gas must follow closed orbits (and in the same sense), but stars in general don't. Two streams of stars can go through each other and hardly notice, but two streams of gas will be shocked (and probably form stars).

PROBLEM 1.1: Prove that if a homogenous sphere of a pressureless fluid with density ρ is released from rest, it will collapse to a point in time $t_{\mathrm{ff}} = 1/4\sqrt{3\pi/(2G\rho)}$ (Binney and Tremaine # 3.4). This is known as the **free-fall time** for the system.

TYPES OF GALAXIES

There are three broad categories of galaxies:

DISK GALAXIES These have masses of $10^6 M_\odot$ to $10^{12} M_\odot$. The disks brightness tend to be roughly exponential, i.e.,

$$I(R) = I_0 \exp[-R/R_0] \tag{1.4}$$

I_0 is $\sim 10^2 L_\odot\,\mathrm{pc}^{-2}$. The scale radius R_0 is $\simeq 4\,\mathrm{kpc}$ for the Milky Way. The visible component is $\simeq 95\%$ stars (dominated by F and G stars for giant spirals), and the rest dust and gas. The more gas-rich disks have spiral arms, and arms are regions of high gas density that tend to form stars; clumps of nascent stars are observed as H II regions. Disk galaxies have bulges which appear to be much the same as small ellipticals. All disk galaxies seem to be embedded in much larger dark halos; the ratio of total mass to visible stellar mass is $\simeq 5$, but we don't really have a good mass estimate for any disk galaxy.

Elliptical galaxies These have masses from $10^6 M_\odot$ to $10^{12+} M_\odot$. There are various functional forms around for fitting the surface brightness, of which the best known is the de Vaucouleurs model

$$I(R) = I_0 \exp\left[-\alpha(R/R_0)^{\frac{1}{4}}\right]. \qquad (1.5)$$

with $I_0 \sim 10^5 L_\odot\,\mathrm{pc}^{-2}$ for giant ellipticals. (To fit to observations, one typically unsquashes the ellipses to circles first. Also, the functional forms are are only fitted to observations over the restricted range in which $I(R)$ is measurable. So don't be surprised to see very different looking functional forms being fit to the same data.) The visible component is almost entirely stars (dominated by K giants for giant ellipticals), but there appears to be dark matter in a proportion similar to disk galaxies. Ellipticals of masses $\lesssim 10^{11} M_\odot$ rotate as fast as you'd expect from their flattening; giant ellipticals rotate much slower, and tend to be triaxial—more on this later.

At the small end of ellipticals, we might put the globular clusters, even though they occur inside galaxies rather than in isolation. These are clusters of masses from $10^4 M_\odot$ to $10^{6.5} M_\odot$, consisting exclusively of very old stars.

Irregulars Everything else! They tend to have strong emission lines, and their starlight is dominated by B,A and F types. Basically, they look like they've just been shaken up and are responding by forming stars.

Hubble types On the whole, galaxy classification probably shouldn't be taken as seriously as stellar classification, because there isn't (yet) a clear physical interpretations of what the gradations mean. But some physical properties do clearly correlate with the so-called Hubble types, so it's worth learning about these at least.

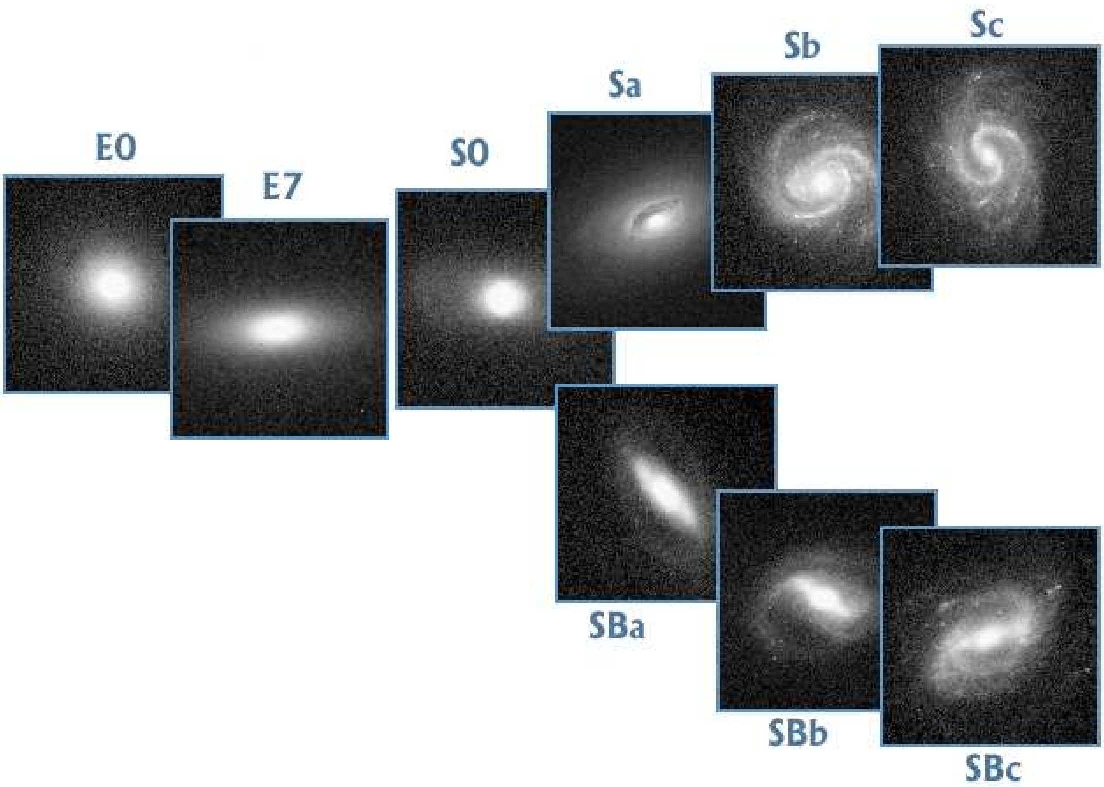

Figure 1.1: The tuning fork diagram of Hubble types

Figure 1.1 shows the Hubble types. Ellipticals go on the left, labelled as En, where $n = 10(1 - \langle \text{axis ratio} \rangle)$. Then the lenticulars or disk galaxies without spiral arms: S0 and SB0. Then spirals with increasingly spaced arms, Sa etc. if unbarred, SBa etc. if barred.

The left ones are called early types, and the right ones late types. People once thought this represented an evolutionary sequence, but that's long been obsolete. (Our current understanding is that, if anything, galaxies tend to evolve towards early types.) But the old names are still used.

We never see ellipticals flatter than about E7. The reason (as indicated by simulations and normal mode analyses) seems to be that a stellar system any flatter is unstable to buckling, and will eventually settle into something rounder.

Note that bulges get smaller as spiral arms get more widely spaced. Theory for spiral density waves predicts that the spacing between arms is proportional to the disk's mass density.

An an evolutionary picture, you should not take the Hubble tuning-form diagram seriously. Hubble thought that all galaxies form as ellipticals, hence he called them "early-types", and that they transform into "late-type" spirals. We now know know that indeed the reverse is true, and the process of this transformation is far from simple.

How galaxies form We can start putting together a general picture now. (The rest of this paragraph varies from well-accepted to controversial to wildly speculative, so don't take it too seriously.) Primordial gas will tend to form rotating disks. Differential rotation in the disks will cause spiral density waves, enhancing density along spiral arms and preferentially forming stars. A bulge-less stellar disk is actually unstable to buckling, and produces a bulge with part of its mass. (That's what simulations indicate.) A bulge formed this way will be rotationally supported like the disk that gave rise to it. Meanwhile the disk will continue to form stars, so disk stars will tend to be younger than a bulge stars. Disks that have turned almost all their gas into stars will have stellar disks, but no spiral arms. Now, a disk galaxy can be disrupted by the gravitational influence of another galaxy. It can be a merger of two or more galaxies, or the tidal disruption of a single galaxy; both tending to disrupt disks and produce irregulars with much star formation, then ellipticals. Disruptions of single galaxies will tend to produce rotationally supported ellipticals; but for mergers the angular momentum vectors will tend to cancel, producing pressure support. So we might expect giant ellipticals to be pressure supported. But even a completely gas-free elliptical will generate gas from its dying stars. This second-generation gas will of course settle into disks, and there we might see spiral arms all over again... And all this while, dark matter (whatever it is) will be finding gravitational potential wells in the neighbourhood of galaxies and form haloes around them.

Note, by the way, that all galaxies appear to have *some* stars $\sim 10^{10}$yr old. Evidently galaxies all formed fairly early, though they have merged or been otherwise disrupted much more recently.

2. Dynamics in a gravitational field

A system of stars behaves like a fluid, but one with unusual properties. In a normal fluid two-body interactions are crucial to its dynamics, but close encounters between stars are very rare. Instead the dynamics of a star can be expressed in terms of its interaction with the mean gravitational field of all the other stars in the system.

Definitions

The gravitational field at a point $\mathbf{x}$, defined as the gravitational force on a unit mass, is given by

$$\begin{aligned} \mathbf{F}(\mathbf{x}) &= G \sum \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \, \Delta m(\mathbf{x}'), \\ &= G \int \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \, \rho\left(\mathbf{x}'\right) d^3\mathbf{x}', \end{aligned} \tag{2.1}$$

in the limit of a continuous medium. A more useful quantity is the gravitational potential $\Phi\left(\mathbf{x}\right)$

$$\Phi\left(\mathbf{x}\right) = -G \int \frac{\rho\left(\mathbf{x}'\right)}{|\mathbf{x}' - \mathbf{x}|} \, d^3\mathbf{x}', \tag{2.2}$$

Problem 2.1: Show, by differentiating (2.2), that the gravitational field is related to the potential by

$$\mathbf{F}(\mathbf{x}) = -\nabla\Phi(\mathbf{x}). \tag{2.3}$$

From the above expression, it is evident that the gravitational field is conservative.

Circular Speed

An important quantity that is often used in spherically symmetric distributions is the *circular speed* $v_c\left(r\right)$, which is the speed a test particle would have in a circular orbit of radius r about the origin of the mass distribution. If $M\left(< r\right)$ be the mass within radius r of a spherical distribution, then

$$\frac{v_c^2\left(r\right)}{r} = -F_r(r) \equiv \frac{d\Phi}{dr} = \frac{GM\left(< r\right)}{r^2}.$$

Thus the circular speed is a measure of the mass inside of r.

A related quantity is the *escape velocity* $v_e\left(r\right)$, which is the speed required to escape to $r = \infty$. Equating the kinetic energy with the gravitational energy of a tiny test mass,

$$v_e\left(r\right) = \left[2|\Phi\left(r\right)|\right]^{\frac{1}{2}}.$$

Only when the speed of a star is greater than this value does its (positive) kinetic energy $\frac{1}{2}mv^2$ exceed the absolute value of its (negative) potential energy, and the star can escape from the gravitational field represented by the potential Φ.

Poisson's equation

Poisson's equation is one of the most useful equations of stellar dynamics:

$$\nabla^2 \Phi(\mathbf{x}) = 4\pi G \rho(\mathbf{x}). \tag{2.4}$$

It is the gravitational analogue of Gauss's law in electrostatics, and can be derived by taking the divergence of (2.1), and applying the divergence theorem. If you are interested, a derivation is available in most books on dynamics (*e.g.* BT §2.1).

Integrating both sides of (2.4) over an arbitrary volume containing mass M, and applying the divergence theorem, we obtain

$$\int \nabla\Phi \cdot d^2\mathbf{S} = 4\pi G M,$$

which can be rephrased as

> Theorem [**Gauss**] *The integral of the normal component of $\nabla\Phi$ over a closed surface is equal to $4\pi G$ times the mass contained within that surface.* □

Spherical systems

The most useful results that enable us to calculate the gravitational field and potential of any spherically symmetric distribution of matter are due to Newton:

> Theorem [**Newton I**] *The net gravitational force exerted by a spherical shell of matter on a particle at a point inside the shell is identically zero.* □

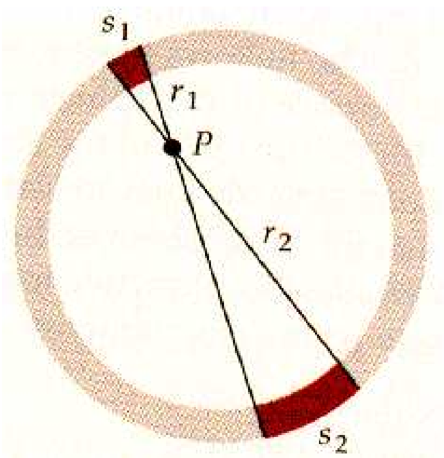

Figure 2.1: Proof of Newton's first theorem.

Figure 2.1 illustrates Newton's first theorem. Consider the cones originating from the point P intersecting the spherical uniform shell of matter at distances r_1 and r_2. The circles of intersection have areas πr_1^2 and πr_2^2 respectively. If the mass per unit area of the shell is σ, it is easy to see that the net gravitational force at P due to these two elements is zero. This argument can be repeated with cones centred at P that intersect the rest of the sphere, and one concludes that the particle at P experiences no net force from the shell.

This implies that the gravitational potential $\Phi(\mathbf{x})$ inside the sphere is constant, since $\mathbf{F}(\mathbf{x}) = -\nabla\Phi(\mathbf{x})$. The easiest place to evaluate it is at the centre, which is equidistant from every point on the sphere, which implies

$$\Phi = -\frac{GM}{R}.$$

THEOREM [**Newton II**] *The gravitational force on a particle that lies outside a closed spherical shell of matter is the same as it would be if all the shell's matter were concentrated into a point at the centre of the shell.* □

The proof is not easy, but is easily found in textbooks. These two theorems enable us to calculate the gravitational potential due to an arbitrary spherically symmetric mass distribution of density $\rho(\mathbf{x})$, which we can split into two parts: the contribution from shells with $\mathbf{r}<\mathbf{x}$ and with $\mathbf{r}>\mathbf{x}$:

$$\Phi(\mathbf{x}) = -4\pi G\left[\frac{1}{x}\int_0^x \rho(r)\, r^2\, dr + \int_x^\infty \rho(r)\, r\, dr\right]. \tag{2.5}$$

Of course the gravitational force will have contributions only from the shells with $r<|\mathbf{x}|$.

PROBLEM 2.2: Show that you can arrive at Eq. 2.5 by integrating the following expression for the potential

$$\nabla\Phi = \frac{G}{r^2}\int_0^r \rho(s)\cdot 4\pi s^2\, ds.$$

Think of the boundary conditions at either extreme of r.

PROBLEM 2.3: Astronauts orbiting an unexplored planet find that (i) the surface of the planet is precisely spherical; and (ii) the potential exterior to the planetary surface is exactly $\Phi = -GM/r$. Can they conclude from these observations that the mass distribution in the interior of the planet is spherically symmetric? If not, can you think of a nonspherical mass distribution that would reproduce the observations? [BT problem 2.1]

A FEW SPHERICALLY SYMMETRIC EXAMPLES

Now we can apply the above results to some simple, and useful, cases.

<u>POINT MASS:</u> For a point mass, the system is analogous to the case of the solar system and is often called the *Keplerian* case:

$$\Phi(r) = -\frac{GM}{r} \quad ; \quad v_c(r) = \left(\frac{GM}{r}\right)^{\frac{1}{2}} \quad ; \quad v_e(r) = \left(\frac{2GM}{r}\right)^{\frac{1}{2}}. \tag{2.6}$$

The circular speed declines with radius as $v_c \propto r^{-1/2}$, which should be the trend far outside any finite mass distribution.

<u>HOMOGENEOUS SPHERE:</u> If the density ρ inside a sphere is constant, then

$$M(r) = \frac{4}{3}\pi r^3 \rho,$$

and the circular velocity is

$$v_c = \left(\frac{4\pi G\rho}{3}\right)^{\frac{1}{2}} r.$$

The circular velocity in this case rises linearly with radius, which means that the angular velocity $\omega \equiv v/r$ is constant. The body in question thus moves like a solid body.

DIGRESSION [Rotation curves of spiral galaxies] When the circular velocity of neutral hydrogen gas was measured well outside the visible limits of spiral galaxies by radioastronomers, it was expected that these velocities would decline in a Keplerian fashion with distance from the centre. Instead, the "rotation curves" of an overwhelming majority of spiral galaxies, representing $v_c(r)$, were found to be almost independent of r out to several times the optical radii of these galaxies. □

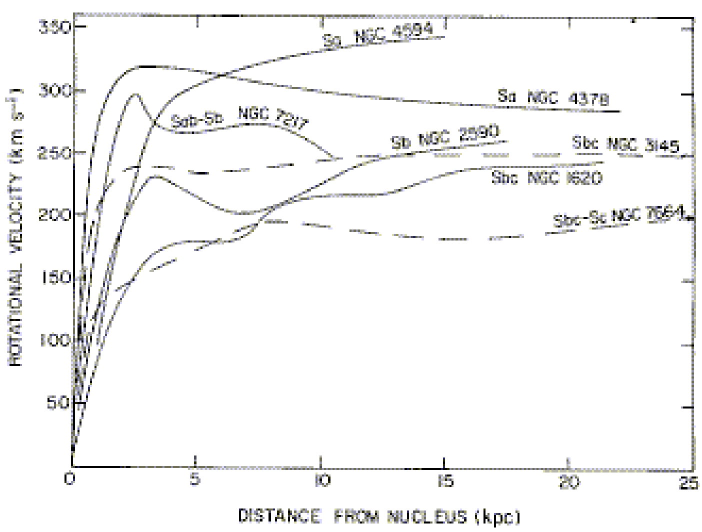

FIG. 3.—Rotational velocities for seven galaxies, as a function of distance from nucleus. Curves have been smoothed to remove velocity undulations across arms and small differences between major-axis velocities on each side of nucleus. Early-type galaxies consistently have higher peak velocities than later types.

Figure 2.2:

PROBLEM 2.4: Assume that the Earth is a sphere of uniform density, through which a diametric tunnel has been dug, passing through the centre. If a test particle is released from rest into this tunnel, show that (a) it takes time $t = (3\pi/16G\rho)^{1/2}$ to reach the centre, and that (b) its motion is simple harmonic. (c) How is this quantity related to the *free-fall time* referred to in Problem 1.2.

There's no friction of course.

POWER-LAW DENSITY: A spherically symmetric system with a density that falls off as some power of the radius

$$\rho(r) = \rho_0 \left(\frac{r}{r_0}\right)^{-\alpha},$$

is singular at the origin if $\alpha > 0$. The corresponding circular velocity is given by

$$v_c^2(r) = \frac{4\pi G \rho_0 r_0^\alpha}{3-\alpha} r^{2-\alpha}.$$

The mass interior to radius r thus is

$$M(r) \equiv \frac{r v_c^2}{G} = \frac{4\pi \rho_0 r_0^\alpha}{3-\alpha} r^{3-\alpha},$$

which means that the mass grows without limit if $\alpha < 3$. But since spiral rotation curves are flat, *i.e.*, v_c=constant, this suggests that within the haloes of disk galaxies, the mass density ρ is proportional to r^{-2} (see below).

However, the escape velocity

$$v_{\rm esc}^2(r) = 2\int_r^\infty \frac{GM(s)}{s^2}\, ds = \frac{2}{\alpha-2} v_c^2(r)$$

is finite as long as $\alpha > 2$. Over the range $3 > \alpha > 2$, the ratio $v_{\rm esc}/v_c$ rises from $\sqrt{2}$ logarithmically to infinity.

PROBLEM 2.5: Find the density $\rho(r)$, circular speed $v_c(r)$ and escape speed $v_{\rm esc}(r)$ for the following model potentials, and find how their Mass behaves as a function of r for large r:

(a) The Hernquist potential

$$\Phi(r) = -\frac{GM}{r+b};$$

(b) The Plummer potential

$$\Phi(r) = -\frac{GM}{\sqrt{r^2+b^2}};$$

(c) The Jaffe potential

$$\Phi(r) = \frac{GM}{b} \ln\left(\frac{r}{r+b}\right),$$

where M and b are constants.

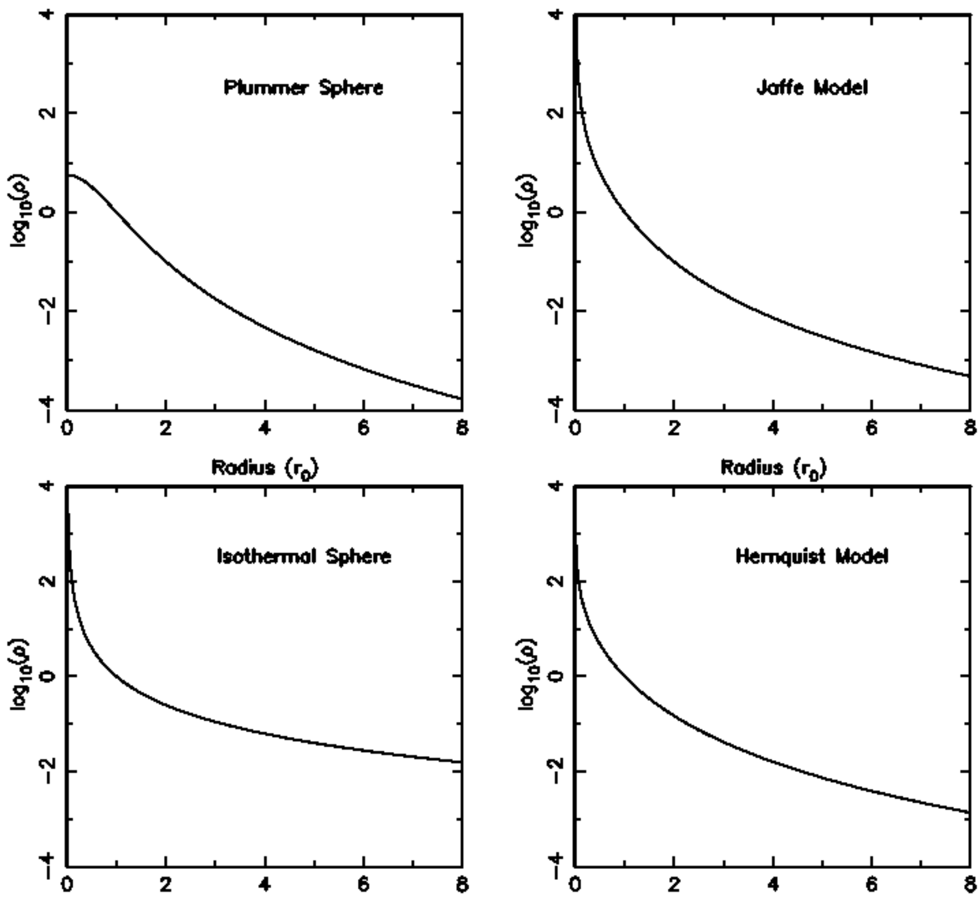

Figure 2.3: Density profiles of common models (from `http://www.astro.utu.fi/~cflynn/galdyn/lecture3.html`).

Table I: Some examples of potential-density pairs

	Potential $\Phi(r)$	Density ρ	v_c^2
Hernquist	$-\frac{GM}{r+a}$	$\frac{M}{2\pi a^3}\frac{a^4}{r(r+a)^3}$	$GM\frac{r}{(r+a)^2}$
Plummer	$-\frac{GM}{\sqrt{r^2+a^2}}$	$\frac{3M}{4\pi a^3}\frac{a^4}{r(r+a)^3}$	$GM\frac{r^2}{(r^2+a^2)^{3/2}}$
Jaffe	$-\frac{GM}{a}\ln\frac{r+a}{r}$	$\frac{M}{4\pi a^3}\frac{a^4}{r^2(r+a)^2}$	$GM\frac{1}{r+a}$

The Plummer density profile has a finite-density core and falls off as r^{-5} as $r \to \infty$, which is a steeper fall-off than is generally seen in galaxies. The Hernquist and Jaffe profiles, on the other hand, both decline like r^{-4} at large r, which has a more sound theoretical basis involving violent relaxation. The Hernquist model has a gentle power-law cusp at small radii, while the Jaffe model has a steeper cusp.

THE SINGULAR ISOTHERMAL SPHERE

We saw above that the rotation curves of almost all spiral galaxies are remarkably flat away from their centre, instead of being the expected Keplerian form (2.6). From the above discussion, this means that at large radii, the mass of a spiral galaxy goes as $M(r) \propto r$ and density $\rho \propto r^{-2}$. This provided an early evidence (in the early 1970s) that the outer parts of galaxies have copious amounts of *Dark matter.* This also means that unless the distribution of matter is cut off at some yet undetermined radius, the mass of each galaxy would diverge. This model density profile is known as the **singular isothermal sphere**.

Unfortunately, the density of such a model diverges as $r \to 0$. In real-life applications, "softened" forms like

$$\rho = \frac{\rho_0}{(1 + r/r_0)^2} \tag{2.7}$$

which have a finite density at the centre (and a "core" of radius r_0) are often used. I invite you to have a look at §4.4(b) of BT.

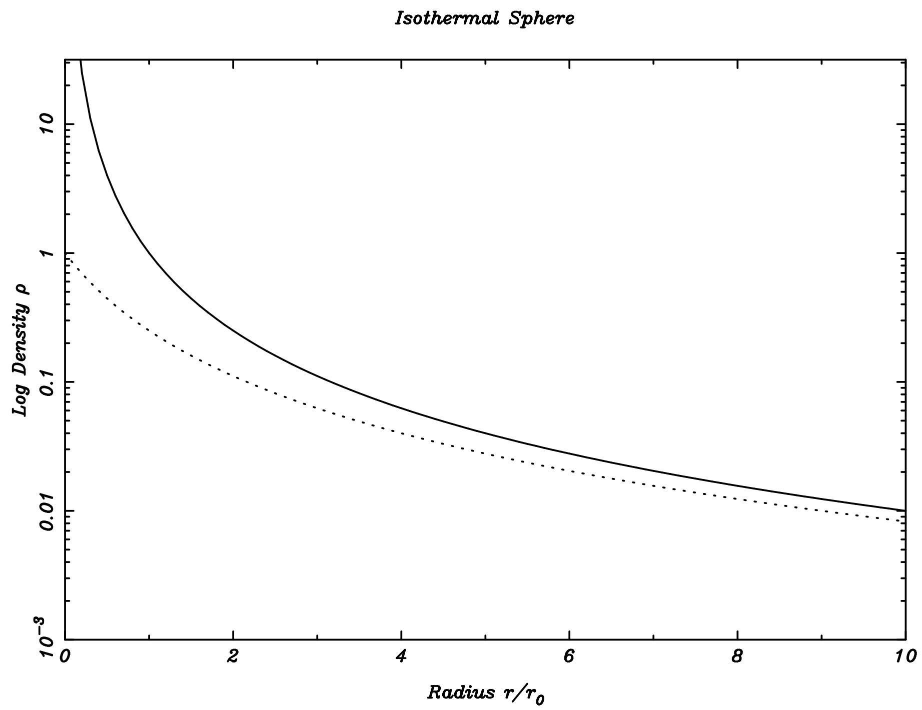

Figure 2.4: The radial dependence of density in the isothermal sphere model. The solid curve represents $\rho \propto r^{-2}$, whereas the dotted curve represents Eq. 2.7.

The virial theorem

Before going into details of stellar orbits, it is worth deriving this basic result that applies to the system of gravitating stars as a whole. In fact it applies to any system of particles bound by an inverse-square force law (e.g. electromagnetism, gravitation), and states that the time-averaged kinetic energy (say $\langle T \rangle$) and the time-averaged potential energy (say $\langle V \rangle$) satisfy

$$2\langle T \rangle + \langle V \rangle = 0. \tag{2.8}$$

To prove this, consider the quantity

$$F = \sum_i m_i \dot{\mathbf{x}}_i \cdot \mathbf{x}_i \tag{2.9}$$

where m_i are the masses. Clearly

$$\frac{dF}{dt} = 2T + \sum_i m_i \ddot{\mathbf{x}}_i \cdot \mathbf{x}_i. \tag{2.10}$$

If F is bounded then the long-time average $\langle dF/dt \rangle$ will vanish. Thus

$$2\langle T \rangle + \sum_i m_i \langle \ddot{\mathbf{x}}_i \cdot \mathbf{x}_i \rangle = 0. \tag{2.11}$$

If the system is gravitationally bound, we have

$$2\langle T \rangle - G \sum_{i \neq j} m_i m_j \left\langle \frac{(\mathbf{x}_i - \mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|^3} \cdot \mathbf{x}_i \right\rangle = 0. \tag{2.12}$$

Interchanging the dummy indices in the second term and adding, we have

$$2\langle T \rangle - \tfrac{1}{2} G \sum_{i \neq j} m_i m_j \left\langle \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|} \right\rangle = 0. \tag{2.13}$$

But the second term is now just minus the total potential energy, which proves the result (2.8).

The virial theorem provides an easy way to makes rough estimates of masses, because velocity measurements can give $\langle T \rangle$. But it is prudent to consider virial mass estimates as order-of-magnitude only, because (i) generally one can measure only line-of-sight velocities using redshift measures from spectra, and getting $T = \frac{1}{2}\sum_i m_i \dot{\mathbf{x}}_i^2$ from there requires more assumptions (e.g. isotropy of the velocity distribution); and (ii) the systems involved may not be in a steady state, in which case of course the virial theorem does not apply. Clusters of galaxies are particularly likely to be quite far from a steady state– we saw why in the previous chapter, in the context of the discussion on crossing times and relaxation times.

Applications of the virial thoerem

1: The Mass of the Perseus cluster of galaxies

The radial velocity dispersion of the Perseus cluster of galaxies is $\sigma_r = 1100$ km/s, and its radius is $2.1\,h_{70}^{-1}$ Mpc, where the Hubble constant is $H_0 = 70\,h_{70}$ km/s/Mpc. Assuming the cluster to be a sphere of uniform mass density ρ, and applying the virial theorem $2T + V = 0$, one can work out its mass in the following steps:

Kinetic Energy (T): Observers measure radial velocities v_r of galaxies from Doppler shifts in their spectra. The mean of all redshifts in a cluster $\langle v \rangle$ would yield the mean radial velocity of the cluster with respect to the observer (largely due to the Hubble expansion of the Universe). The dispersion σ_r^2 of the measured values of v_r about this mean would be a measure of the K.E. of the galaxies, so

$$T = 3 \times \frac{1}{2}\,M\sigma_r^2,$$

where M is the combined mass of all the galaxies, the factor 3 accounting for the fact that one measures only the radial component of the velocities of the galaxies, whereas the kinetic energy would depend on the net spatial velocity v of each galaxy, and statistically $\langle v^2 \rangle = 3\langle \sigma_r^2 \rangle$.

Potential Energy (V): The potential energy of a uniform sphere is calculated from the work done to assemble the sphere out of shells of matter brough in from infinity (work it out yourself!). Since gravitation is attractive, this quantity would be negative, and is turns out that for a sphere of radius R and mass M, one gets

$$V = -\frac{3}{5}\frac{GM^2}{R}. \tag{2.14}$$

You would get the same answer for the potential energy of a sphere of uniform positive charge due to electrostatic forces, but the sign would be positive.

So, one can estimate the *virial mass* $M = 5\,R\langle \sigma_r^2 \rangle / G$, given the radius of the cluster and its radial velocity dispersion.

Problem 2.6: Derive Eq. 2.14 for the potential energy of a gravitating sphere. Consider the work done to assemble the sphere from thin shells brought from an infinite distance away.

Problem 2.7: Zwicky (1933) first pointed out, using the virial theorem, that there was 400 times as much dark matter as luminous matter in the Coma cluster of galaxies. However, his conclusion was based on a Hubble constant of $H_0 = 558$ km/s/Mpc. How would his conclusion about the ratio of dark to luminous matter (known formally as mass-to-light ratio, M/L) be affected, if we believe in the currently popular value of the Hubble constant $H_0 = 70$ km/s/Mpc?

2: THE TEMPERATURE OF THE INTERGALACTIC GAS IN A SPHERICAL GALAXY

Imagine a spherical galaxy forming from a collapsing cloud of gas. A fraction of the gas has turned into stars, but some of it is left over in the system as intergalactic gas, in virial equilibrium with the gravitational potential of the entire galaxy, dark matter and all. If we imagine this interstellar medium (ISM), which is mostly hydrogen, to be an ideal gas, its mean square velocity $\langle v^2 \rangle$ is found by equating

$$\frac{1}{2}\mu m_p \langle v^2 \rangle = \frac{3}{2} k_B T_{\rm vir},$$

where μm_p is the mean mass of each particle of the gas (for instance, if it is pure atomic hydrogen, $\mu = 1$, but if it is ionized pure hydrogen, then there are twice as many particles, but only half of them are far more massive compared to the others, so $\mu = 0.5$), k_B is the Boltzmann constant, and $T_{\rm vir}$ is its *virial temperature*. One can also assume that the gas is in dynamical equilibrium with the galaxies, so if $\langle v^2 \rangle$ is the rms speed of the gas particles of the ISM, then, as in the previous section, $\langle v^2 \rangle = 3\langle v_r^2 \rangle$, where $\langle v_r^2 \rangle$ is the radial velocity dispersion measured from the redshifts of galaxies. Furthermore, from the above example, $\langle \mathbf{v}^2 \rangle = 3GM/5R$. Therefore, the virial temperature of the gas is

$$T_{\rm vir} = \frac{GM}{5R}\frac{\mu m_p}{k_B} = 1 \times 10^6 \mu \left(\frac{M}{10^{11}\, M_\odot} \right) \left(\frac{10\,{\rm kpc}}{R} \right) {\rm K}.$$

One can easily see that the hot interstellar medium (ISM) of galaxies or the intergalactic (IGM) medium of clusters of galaxies is expected to emit mostly in the X-ray region of the electromagnetic spectrum.