Potential-Density pair: Worked Problem

In a spherical galaxy, the density of matter varies with radius as

$$\rho\,(r) = \frac{M}{4\pi}\frac{a}{r^2(r+a)^2},$$

where M and a are constants.

(a) Verify that the total mass of the galaxy is M.

$$\text{Total Mass} = \int_0^\infty 4\pi r^2 \rho dr = \int_0^\infty \frac{Ma}{(r+a)^2}dr = Ma\left[-\frac{1}{r+a}\right]_0^\infty = M.$$

(b) Show that the gravitational potential $\Phi\,(r)$ of the above galaxy as a function of radius r is

$$\frac{GM}{a}\ln\frac{r}{r+a}.$$

Use Poisson's Equation in spherical coordinates. Since the problem is spherically symmetric, only the d/dr terms are relevant.

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G\rho$$

$$r^2\frac{d\Phi}{dr} = \int_0^r GMa\frac{dr}{(r+a)^2} = -GMa\left[\frac{1}{r+a}\right]_0^r = GM\frac{r}{r+a}.$$

$$\frac{d\Phi}{dr} = GM\frac{1}{r(r+a)} = \frac{GM}{a}\left(\frac{1}{r} - \frac{1}{r+a}\right).$$

$$\Phi = \int_0^r \frac{GM}{a}\left(\frac{1}{r} - \frac{1}{r+a}\right)dr = \frac{GM}{a}\left(\ln r - \ln(r+a)\right)dr = \frac{GM}{a}\ln\frac{r}{r+a}.$$

(c) Show that for large radii $r \gg a$, the potential approaches that of a point mass.

At large radii $r \gg a$, $a/r \to 0$. Remember for small x, $\ln(1+x) \approx x$.

$$\Phi = \frac{GM}{a}\ln\frac{r}{r+a} = \frac{GM}{a}\ln\left(1+\frac{a}{r}\right)^{-1} = -\frac{GM}{a}\ln\left(1+\frac{a}{r}\right) \approx -\frac{GM}{r},$$

which is the potential for a point mass.

(d) Find an expression for the circular velocity, and show that it is approximately constant at small radius ($r \ll a$) and find its radial dependence at large radii ($r \gg a$).

$$v_c^2 = r\left|\frac{d\Phi}{dr}\right| = \frac{GM}{r+a}.$$

For $r \ll a$, this reduces to $v_c\,(r) \propto \frac{GM}{a}$, which is independent of r.

For $r \gg a$, this reduces to $v_c\,(r) \propto \frac{GM}{r}$, which is the Keplerian dependence $v_c \propto r^{-1/2}$.

3. Stellar orbits

The usual approach to modelling galaxies is to look for a way of combining a realistic potential with a distribution of stars following possible orbits within the potential, such that it self-consistently provides the mass density distribution that gives rise to the potential we considered in the first place. Now we'll consider the nature of orbits in potential models of the kind discussed in the previous section.

Integrals and constants of motion

The motion of any particle can be described by its location in phase space, given by the six quantities $\{\mathbf{x}(t),\ \mathbf{v}(t)\}$.

- An **integral of motion** $I(\mathbf{x}, \mathbf{v})$ is a function of phase space coordinates that remains constant along any orbit.
- A **constant of motion** $C(\mathbf{x}, \mathbf{v}, t)$ is a function of phase space coordinates *and time* that remains constant along any orbit.

Every integral is a constant of motion, but not all constants are integrals of motion.

For instance, if we consider the motion of a star in a static potential, then the energy per unit mass $\Phi + \frac{1}{2}v^2$ is both an integral and a constant of motion. For a star moving in a spherically symmetric potential, the energy and all the components of the angular momentum vector are integrals of motion. On the other hand, in an axisymmetric potential, only the component of the angular momentum along the axis of symmetry is an integral of motion. For a planet orbiting a star, the phase of the orbit, given by

$$\phi = \phi_0 + \int \frac{L}{r(t)^2} dt,$$

where ϕ_0 is a constant and L the orbital angular momentum, is a constant of motion, but not an integral.

The two-body central force problem

The problem of two bodies of mass m_1 and m_2 moving under the influence of a mutual central force can be reduced to the equivalent one-body problem, where a single body of mass μ orbits a fixed center of force (the centre of mass), where

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}.$$

Consider the orbit of a star that moves under the influence of a conservative central force, *i.e.* the potential $V(r)$ is a function of r only, and the force is always along $\mathbf{r}$. Since this is a spherically symmetric system, the total angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is conserved. Therefore, $\mathbf{r}$ is always perpendicular to a fixed direction $\mathbf{L}$ in space, which means that $\mathbf{r}$ is always in a plane normal to $\mathbf{L}$. This is of course unless $L = 0$, in which case the motion must be along a straight line passing through the origin (the centre of the force), and $\mathbf{r}$ is parallel to $\mathbf{v}$, which is a straight line orbit. Thus the motion of a particle under the influence of a central force is always in a plane. Here, we are going to consider this to be the (r, θ) plane and will ignore any variation in the ϕ component.

In spherical polar coordinates, if $\mathbf{r} = r\,\hat{r} + \theta\,\hat{\theta} + \phi\,\hat{\phi}$, then its derivatives are

$$\begin{aligned} \dot{\mathbf{r}} &= \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta} + r\sin\theta\,\dot{\phi}\,\hat{\phi} \\ \ddot{\mathbf{r}} &= (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2)\,\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta} - \tfrac{1}{2}r\sin 2\theta\,\dot{\phi}^2)\,\hat{\theta} + (\ldots)\,\hat{\phi} \end{aligned} \tag{3.1}$$

Therefore, in the central force problem, the equations of motion are

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= F(r)/m, \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} &= 0 \end{aligned} \quad , \tag{3.2}$$

where m is the mass of the particle.

Multiplying the second of (3.2) by $\mathbf{r}$ on both sides, and integrating with respect to t, we get the familiar integral of motion

$$r^2\dot{\theta} = \text{constant } (\equiv L/m), \tag{3.3}$$

where L is the (conserved) angular momentum. From this it also follows that the area swept out by the line joining the two bodies per unit time, which is given by $\frac{1}{2}r^2\dot{\theta}$ is conserved, yielding a general form of Kepler's second law. Note that this result holds for *all* central forces, not just the r^{-2} case as in the Kepler problem.

We can substitute (3.3) into the first of (3.2) to yield an equation involving r and its derivatives only:

$$m\ddot{r} - \frac{L^2}{mr^3} = F(r). \tag{3.4}$$

This is the same equation that would be obtained for a fictitious one-dimensional problem in which a particle of mass m is subject to a force

$$F'(r) = F(r) + \frac{L^2}{mr^3}. \tag{3.5}$$

The significance of the additional term is clear if it is written as

$$mr\dot{\theta}^2 = mv_\theta^2/r,$$

the familiar centrifugal force. The corresponding potential is given by

$$V'(r) = V(r) + \tfrac{1}{2}\frac{L^2}{mr^2}. \tag{3.6}$$

We will call this the *effective potential*. Furthermore, the energy conservation relation implies that the total energy

$$E \equiv V'(r) + \tfrac{1}{2}m\dot{r}^2 = V(r) + \tfrac{1}{2}\frac{L^2}{mr^2} + \tfrac{1}{2}m\dot{r}^2 \tag{3.7}$$

is constant.

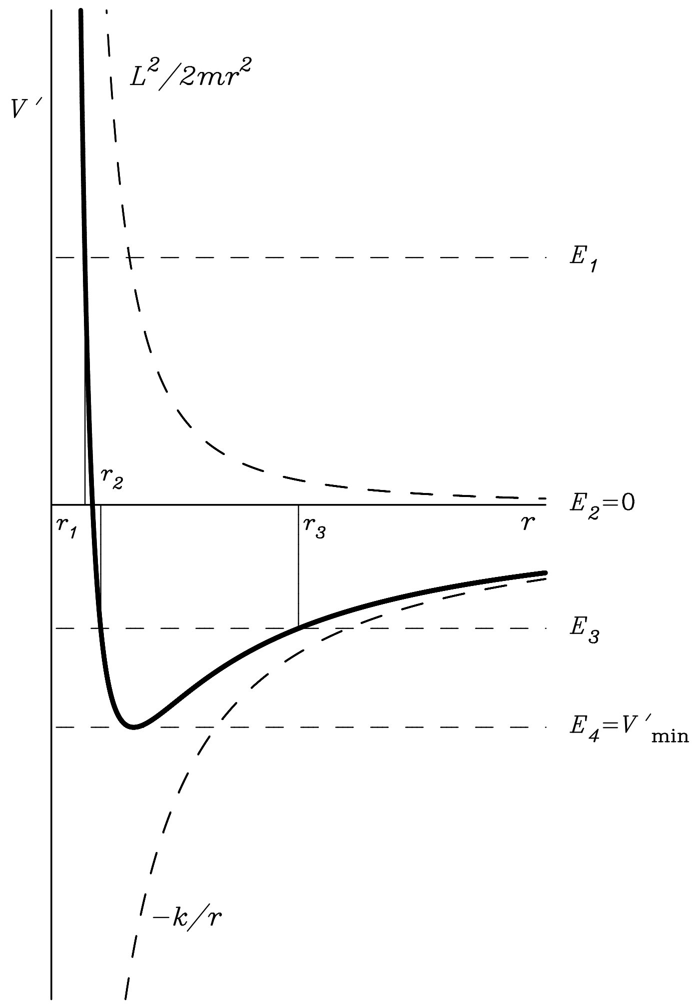

Figure 3.1: The equivalent one-dimensional "effective" potential for an attractive inverse-square central force.

The inverse-square central force

For a inverse-square central force, *e.g.* gravitation,

$$F(r) = -\frac{k}{r^2}, \qquad V(r) = -\frac{k}{r}, \tag{3.8}$$

and the corresponding effective potential is

$$V'(r) = -\frac{k}{r} + \frac{L^2}{2mr^2}.$$

This quantity is plotted in Figure 3.1 as a function of r, where the two dashed lines represent the first and second terms on the right-hand side respectively, and the solid line is the sum $V'(r)$.

Consider the motion of a particle with energy E_1, as shown in the Figure, such that $E_1 \gg 0$. In this case, if $r < r_1$, then $V' > E_1$, and the kinetic energy $\frac{1}{2}m\dot{r}^2$ in (3.7) will have to be negative, which is not possible since it involves the square of the velocity. This means that a star of energy E_1 can never come closer than r_1 to the centre. Since the high positive value of E_1 is due to the angular momentum L, it follows that in a two-body system with substantial angular momentum, neither of the particles can pass through the centre in their orbit. This explains why, for example, the Earth does not fall into the Sun even though the attraction between the two is entirely radial.

For a particle with energy $E_3 < 0$, in addition to a lower bound r_2, there is also a maximum value r_3 that cannot be exceeded with positive kinetic energy. Stars with this energy will thus be bounded, their orbit always lying between $r_2 < r < r_3$. If the energy is E_4 just at the minimum of the effective potential V', then these two bounds coincide, which means that motion is possible at only one value of r, *i.e.* the orbit is a circle. This occurs when F' is zero, *i.e.* when $F(r) = -mr\dot{\theta}^2$. This is the familiar case where the applied force is equal and opposite to the "reversed effective force" of centripetal acceleration. Finally a star with $E < E_4$ cannot have a feasible orbit, and will free-fall into the centre.

PROBLEM 3.1: The above exercise is valid for an inverse-square force (3.8). For other forces, the orbits may not have such simple forms, though the same general division into open, bounded and circular orbits will be true for any attractive potential that (a) falls off slower than r^{-2} as $r \to \infty$, and (b) becomes infinite slower than r^{-2} as $r \to 0$. Verify this statement.

PROBLEM 3.2: Sketch a figure similar to Figure 3.1 for the one-dimensional central force $F = -kx^{-4}$ and find the different kinds of orbits possible for all possible ranges of values of E and r.

DIGRESSION [Another central force: Hooke's law] Another interesting example occurs when the linear central force is proportional to r, such that

$$F(r) = -kr, \qquad V(r) = \tfrac{1}{2}kr^2. \tag{3.9}$$

This corresponds to the isotropic harmonic oscillator, where the restoring force is the same as that in a spring, for instance. The motion under this force is bounded for all possible energies and does not pass through the centre, unless $L = 0$, when the motion is simple harmonic along a straight line. For $L \neq 0$, the orbit is elliptical with the centre of the force is at the centre of the ellipse, not at a focus as in the Kepler case. It is interesting to note that closed repeated orbits are produced in only two cases of central forces, in this case and in the inverse-square case (look up the proof of *Bertrand's theorem* in any advanced text of classical mechanics if you're interested). □

NATURE OF ORBITS

We will illustrate the general nature of orbits by considering the case of spherical symmetry. If, for instance, the nature of the central force is of the form $\Phi(r) = -GM(r^{-1} + ar^{-2})$, the resultant orbit can be represented by

$$\theta = \theta_0 \pm k \arccos\left[\frac{1}{C}\left(\frac{1}{r} - \frac{GMk^2}{L^2}\right)\right] + 2nk\pi, \tag{3.10}$$

where $k \equiv (1 - 2GMa/L^2)^{-\frac{1}{2}}$, and $\arccos(x)$ is the value of Arccos (x) that lies between 0 & π. If k is irrational (most real numbers are), then by choosing a suitable integer n, θ can be made equal to any number we please (modulo 2π). Thus for a given E, L and r, we can have any θ, and so eventually all points in space will be covered by this orbit. On the other hand, if k is rational (like in the Kepler case 3.8, where $a = 0$ and $k = 1$, the orbit closes on itself, since θ can have only a few fixed values for a given E, L and r (two in the Kepler case). Integrals like θ for irrational k that fail to confine orbits are called non-isolating integrals, and are of no value for galactic dynamics.

Orbits in spherically symmetric potentials

Let us illustrate with a couple of specific simple examples. The basic set of equations of motion are (3.2) and (3.3). Writing the specific angular momentum as $\ell \equiv L/m \equiv r^2\dot{\theta} =$ constant, and the acceleration as $f \equiv F(r)/m$, the first of (3.2) becomes

$$f = \frac{\ell^2}{r^2}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) - \frac{\ell^2}{r^3}. \tag{3.11}$$

Putting $u \equiv 1/r$, one obtains the familiar equation

$$\frac{d^2u}{d\theta^2} + u = \frac{f(1/u)}{\ell^2u^2}. \tag{3.12}$$

Solutions to this equation can be of two types: unbound orbits where as time progresses, $u \to 0$ (or $r \to \infty$). We are not interested in such orbits, since they do not constitute galaxies. Bound orbits, on the other hand, are such that r (and u) oscillate with time between definite bounds. Multiplying (3.12) by $du/d\theta$ and integrating, one gets the energy equation

$$\left(\frac{du}{d\theta}\right)^2 + \frac{2\Phi}{\ell^2} + u^2 \equiv \frac{2E}{\ell^2} \tag{3.13}$$

where the constant of integration has been written on the right-hand side in terms of the total energy per unit mass E and specific angular momentum ℓ. If you are wondering where the $\Phi(r)$ came from, recall $f(r) = -d\Phi(r)/dr$. For the bound orbits, at the limits of u, the quantity $du/d\theta$ vanishes, so the two extremes are given by the roots of the quadratic equation

$$u^2 + \frac{2\left[\Phi(\frac{1}{u}) - E\right]}{\ell^2} = 0, \tag{3.14}$$

between which the star will oscillate. The inner radius $r_1 = 1/u_1$ is called the *pericentre*, and the outer radius $r_2 = 1/u_2$ the *apocentre*.

Radial and Azimuthal periods The energy equation above, (3.13) can be rewritten as

$$E = \Phi + \tfrac{1}{2}\dot{r}^2 + \tfrac{1}{2}(r\dot{\theta})^2,$$

from which $\dot{\theta}$ can be eliminated using (3.3), giving

$$\frac{dr}{dt} = \pm\sqrt{2(E-V) - \frac{\ell^2}{r^2}}.$$

The $\pm$ signs indicate that there are two cases where the star alternately moves towards the centre and away. From (3.14) you can verify that $\dot{r} = 0$ at the pericentre and apocentre of the orbit. The radial period is the time spent in travelling from pericentre to apocentre and back to pericentre,

$$T_r = 2\int_{r_1}^{r_2} \frac{dr}{\sqrt{2(E-V) - \frac{\ell^2}{r^2}}}.$$

Likewise the azimuthal period can be defined as the time taken for the star to go through a whole cycle for the other coordinate $\theta = 2\pi$ in the plane of the orbit (see e.g. Binney

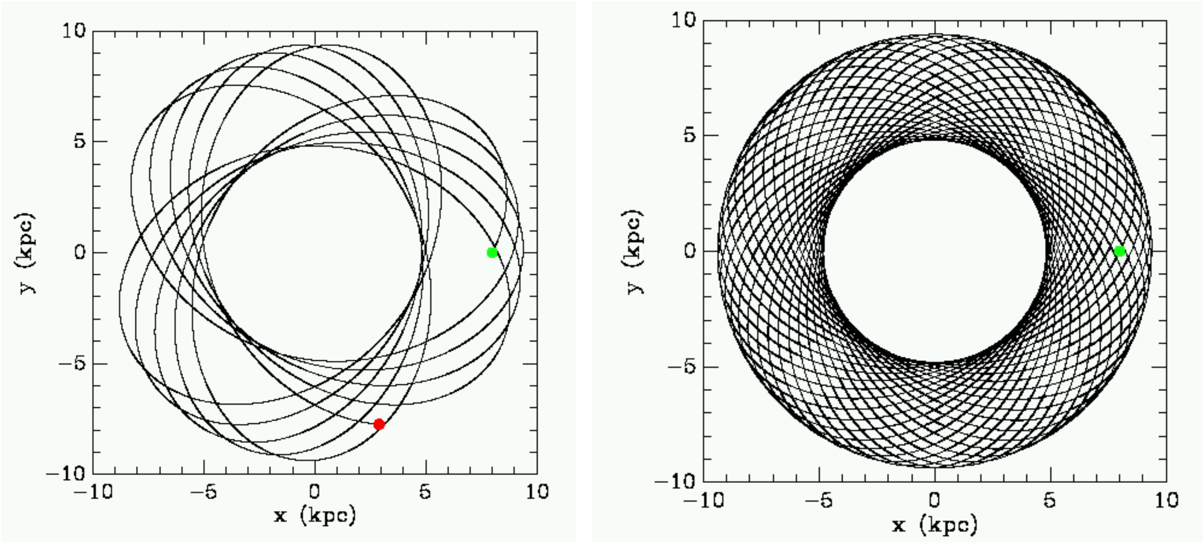

Figure 3.2: The orbit of a star in the potential of the Milky way, given by a spherically symmetric potential that produces a flat rotation curve. A star starts off from the point at $x = 8$ kpc, $y = 0$ kpc, where the Sun is now, with velocity bf v= (90, 180, 0) km/s. Note that it goes off to the right and top since the x and y components of its initial velocity are positive. The left plot shows its motion over 2 Gyr, and the right one, after 20 Gyr.

and Tremaine for details). For a Kepler orbit, for instance, this ratio $T_\theta/T_r = 1$, and the planet, travelling around the Sun, goes over and over the same elliptical orbit as a result. The ratio of the two periods T_θ/T_r is in general not a rational number, and so the typical orbit of a star in a spherically symmetric potential will be a rosette bound between two concentric circles of radius r_1 and r_2, where the star passes through every point between the two circles, given enough time.

Clearly these orbits, which do not pass through the centre, cannot be the only constituents of galaxies, which don't look like doughnuts. But they could easily make up the disks of spiral galaxies. We will look at other kinds of orbits in our discussion of axisymmetric potentials later on. Meanwhile, let's consider two special cases of orbits in spherially symmetric potentials.

The Spherical harmonic oscillator Consider the spherically symmetric potential

$$\Phi(r) = \tfrac{1}{2}\omega^2 r^2 + \text{constant},$$

which, as you would recall, we encountered in the case of a sphere of constant density, where the circular velocity $v_c^2(r) \propto r^2$. We needn't derive the equation for u in this case. Simply writing down the solution in Cartesian coordinates (which is possible in this case) would help us see clearly what is going on. In Cartesian coordinates, $x = r\cos\theta$ and $y = r\sin\theta$, and the equations of motion in these two directions reduce to $\ddot{x} = -\omega^2 x$ and $\ddot{y} = -\omega^2 y$, where $\omega^2 = 4\pi G\rho/3$. These are equations of simple harmonic motion (remember that in the problem of the particle in a tunnel through the constant-density Earth, we were dealing with one of these dimensions). The solutions thus are

$$x = X\cos(\omega t + \epsilon_1); \quad y = Y\cos(\omega t + \epsilon_2).$$

This orbit is elliptical in general (circular if $X = Y$), with the centre of attraction at the centre of the ellipse. This of course is one of two cases, cited in *Bertrand's theorem*, that yield closed bounded orbits. Let's look at the other such orbit.

The Kepler orbit This is the case where all the mass in the system M is enclosed within the orbit of the star, which is at a distance r from the centre of attraction, and the potential is spherically symmetric (the same case as for a planet in the Solar system). You should refer to more details to any standard textbook on classical mechanics, but here let me write down a few results for comparison.

The force per unit mass on the star is $f(r) = -GM/r^2 = -GMu^2$, so (3.12) can be rewritten as

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{\ell^2}. \tag{3.15}$$

Since $u = 1/r$, the solution can be written as,

$$r(\theta) = \frac{a(1-e^2)}{1 + e\cos(\theta - \theta_0)},$$

where the eccentricity of this conic section orbit is $e = C\ell^2/GM$, C being the constant of integration from (3.15), and the semi-major axis $a \equiv \ell^2/GM(1-e^2)$. For bound orbits, which we are interested in here, the eccentricity $e < 1$, and r is finite for all value to the azimuthal angle θ, and is a periodic function in 2π. These orbits, unlike the previous case, are ellipses with the centre of attraction at one of the foci of the ellipse. The pericentre and apocentre lie at $r_1 = a(1-e)$ and $r_2 = a(1+e)$.

Most often, instead of expressing the radius r as a function the angle θ, you would want to know how it behaves with time t. The bad news is that in general this cannot be written down in closed form in a single equation. This is usually represented as a set of parametric equations, in terms of an angular parameter η,

$$r = a(1 - e\cos\eta); \qquad t = \frac{T_r}{2\pi}(\eta - e\sin\eta),$$

which we will come back to later on. The radial and azimuthal periods, as mentioned above, are equal in this case

$$T_r = T_\theta = 2\pi\sqrt{\frac{a^3}{GM}}.$$

4. Spiral galaxies

The characteristic feature of spiral galaxies is that they have a disk-like appearance with well-defined spiral arms emanating from their central regions. They often have central bars and/or rings. Very often the spiral pattern has a remarkable degree of symmetry with respect to the centre of the galaxy. The light distribution of a 'normal' spiral galaxy is made up of (a) a *central bulge* or *spheroid*, similar to an elliptical galaxy, (b) a disk component, in which the spiral arms lie.

The luminosity of the disk of the spiral can be represented as

$$L(R, z) = L_0 \exp\left(-\frac{R}{R_0}\right) \mathrm{sech}^2\left(\frac{z}{z_0}\right),$$

where R and z are distances measured in the radial direction from the centre of the galaxy and perpendicular to the disk respectively. The gas and dust have scale heights smaller than that of stars. In contrast, the luminosity of the bulge follows a de Vaucouleurs profile (see *Ellipticals*). The central surface brightness of spirals is found to be more or less constant in all spirals (21.65 ± 0.3 mag/sec^2, Freeman's law). It isn't clear whether this is due to a selection effect.

The winding dilemma

Spiral arms are seen in almost all disk galaxies, but they do not appear to be very tightly wound up, even though the disk is rotating. If the disk of a spiral rotates with an angular speed $\Omega(R)$, and at any given epoch a radial stripe is drawn across its disk, the equation for the stripe can be written as

$$\phi\,(R, t) = \phi_0 + \Omega\,(R)\,t. \tag{4.1}$$

The pitch angle i is defined as the angle between the tangent to the arm and the circle r =constant, which is given by

$$\cot i = \left| R \frac{d\Phi}{dr} \right|. \tag{4.2}$$

Imagine that due to the rotation of the disk, the spiral arms are being wound up. If the nearest successive location of arms at azimuth ϕ are at R and $R + \Delta R$, then

$$2\pi = \left| R \frac{d\Phi}{dr} \right|, \tag{4.3}$$

since one winding corresponds to the change in ϕ of 2π. If $\Delta R \ll R$ (i is very small), $\Delta R = 2\pi R / \cot i$. For typical values of a star at the position of the sun in a spiral disk like ours, $v_c = 220$ km/s, $R = 10$ kpc, $t = 10^{10}$ yr, we find $i = 0.5$ degrees and $\Delta R = 0.28$ kpc. This obviously corresponds to a spiral arm that is far too tightly wound that is observed in the real world. Observed pitch angles are about 10 degrees or so, and we've never seen a spiral in which arms are wound up more than 10 times around.

The most likely implication is that spiral arms are not material features. The winding dilemma arises from thinking of spiral arms as material alignments in a differentially rotating disk. The way out of this dilemma is that the spiral structure is a (density) wave phenomenon, maintained by the self-gravity of the distribution of matter in the disk, so that at different times, the density enhancement seen at a given place are is up of different stars/clouds.

Theories of spiral structure

That rotating disk galaxies should exhibit spiral structure isn't surprising, but the nature of spiral structure isn't completely understood. Spiral patterns in disk galaxies can arise from various sources. Kinematic spiral waves are perturbations that naturally arise in a differentially rotating system. Spiral waves can also be caused by tidal interaction with neighbours.

Water molecules in the ocean do not move very far in response to a passing wave. Similarly, stars in a disk galaxy need not move far from their unperturbed orbits to create a spiral density wave. In disk galaxies, most stars are on nearly circular orbits, so it is useful to consider their orbits as basically circular, but with small perturbations in the R and z directions.

Epicycles:

Recall our discussion on the Effective potential in case of central forces. Let's consider a spiral galaxy to be an axisymmetric system, and write the general equation of motion of a star in such a galaxy in cylindrical polar coordinates. The three components of acceleration are given by

$$\begin{aligned} \ddot{R} - R\dot{\phi}^2 &= -\frac{\partial \Phi}{\partial R}, \\ \frac{d}{dt}\left(R^2\dot{\phi}\right) &= 0, \\ \ddot{z} &= -\frac{\partial \Phi}{\partial z}. \end{aligned} \tag{4.4}$$

The second of these equations as usual provides us with the law of conservation of angular momentum $L_z \equiv R^2\dot{\phi}$ =constant.

A star with angular momentum L_z can follow an exactly circular orbit only at the radius R_g where the effective potential is stationary with respect to R, such that

$$\frac{d\Phi_{\rm eff}}{dR} = 0 = \frac{\partial}{\partial R}\left[\Phi\left(R,z\right) + \frac{L_z^2}{2R^2}\right].$$

At $R = R_g$, $z = 0$ (in the meridional plane), this means

$$\left.\frac{\partial \Phi}{\partial R}\right|_{R_g, z=0} = -\frac{\partial}{\partial R}\left(\frac{L_z^2}{2R^2}\right) = \frac{L_z^2}{R_g^3} = R_g\,\Omega^2(R_g),$$

since $\Omega \equiv \dot{\phi}\left(R_g\right) = L_z/R_g^2$ from its definition.

If $\Phi_{\rm eff}$ is minimum at $R = R_g$, the corresponding circular orbit has minimum energy for given L_z, and so is stable. Any star with same L_z will oscillate about this mean orbit, with small perturbations in the radial and z directions. As the star moves radially in and out, its azimuthal motion must alternately speed up and slow down respectively. Therefore such a star would follow an approximately elliptical "epicycle" around the ***guiding centre*** R_g, which moves with angular speed $\Omega(R_g)$ in a circular orbit.

If x and y be coordinates in a 'not-quite-Cartesian' frame of reference revolving about the centre of the galaxy with angular velocity Ω of a circular orbit at radius $R = R_g$. In terms of polar coordinates in the plane of the disk,

$$x \equiv R - R_g; \qquad y \equiv R_g(\phi - \Omega t),$$

where x increases outward from the centre and y increases in the direction of rotation.

The equation of motion in the radial direction is given by

$$\begin{aligned} \frac{d^2}{dt^2}(R_g + x) &= -\frac{\partial \Phi_{\rm eff}}{\partial R} \\ &= -\left[\frac{\partial^2}{\partial R^2}\Phi_{\rm eff}\right]_{R_g}(R_g + x), \end{aligned} \tag{4.5}$$

using Taylor expansion, neglecting higher powers of the small perturbation x. This leads to

$$\ddot{x} = -\left(\frac{\partial^2}{\partial R^2}\Phi_{\rm eff}\right)x = -\kappa^2(R_g)\,x,$$

which is the equation for a simple harmonic solution

$$x = X\,\cos(\kappa t + \psi). \tag{4.6}$$

When $\kappa^2 > 0$, this equation describes a simple harmonic motion with frequency κ. If $\kappa^2 < 0$, the orbit is unstable, and the star moves away from the centre of force.

Since L_z is conserved, and the angular speed of the corresponding circular orbit $\Omega = L_z/R_g^2$, one can write the rate of change of the azimuthal angle of the star as

$$\begin{aligned} \dot{\phi} &= \frac{L_z}{R^2} \\ &= \frac{L_z}{R_g^2}\left(1 + \frac{x}{R_g}\right)^{-2} \\ &\simeq \Omega\left(1 - \frac{2x}{R_g}\right). \end{aligned} \tag{4.7}$$

Substituting from (4.6), we have

$$\dot{\phi} = \Omega\left[1 - \frac{2X}{R_g}\cos(\kappa t + \psi)\right].$$

Integrating with respect to t,

$$\phi = \Omega t + \phi_0 - \frac{2\Omega X}{\kappa R_g}\sin(\kappa t + \psi),$$

such that the tangential displacement

$$\begin{aligned} y &= -\frac{2\Omega X}{\kappa}\sin(\kappa t + \psi) \\ &= -Y\,\sin(\kappa t + \psi). \end{aligned} \tag{4.8}$$

Eqs. (4.6) and (4.8) together give the complete solution to the orbit of the epicycle the star describes in the plane of the galaxy over and above its circular orbit. If $X \neq Y$, this epicycle is elliptical, with ratio of axes

$$\frac{X}{Y} = \frac{\kappa}{2\Omega}.$$

For a harmonic oscillator potential (uniform density, solid body rotation), $X/Y = 1$. However, for a Kepler potential, $X/Y = 1/2$. In general, $Y \geq X$, so that the epicycle ellipse in elongated in the tangential direction.

PROBLEM 4.1: Show that the epicyclic frequency $\kappa\,(R)$ is related to the Oort constant B by

$$\kappa^2 = \frac{1}{R^3}\frac{d}{dR}\left[(R^2\Omega)^2\right] = -4B\Omega,$$

where Ω is the angular speed of the guiding centre. For the Sun, the value of B is negative-what does this signify for the Sun's orbit?

From the expression given in the problem above, it is easy to see that for a Kepler case (point mass at the centre), $\kappa = \Omega$, whereas in the case of solid-body rotation, $\kappa = 2\Omega$. The potential of our Galaxy is in between these two cases, so at the position of the Sun,

$$\Omega < \kappa < 2\Omega.$$

In fact, the measured value at the Sun shows $\kappa \sim 1.4\Omega$. Thus the Sun and nearby disk stars are moving on epicycles which are squashed by about 30% in the radial direction (see BT, §3.2.3). The orbit of the Sun thus does not close on itself, and the period of the epicycle is 170 Myr.

The motion in the perpendicular direction has a similar solution

$$z = Z\,\cos(\nu t + \phi),$$

where the *vertical frequency* ν is given by

$$\nu^2 \equiv \left(\frac{\partial^2}{\partial z^2}\Phi_{\text{eff}}\right)_{z=0}.$$

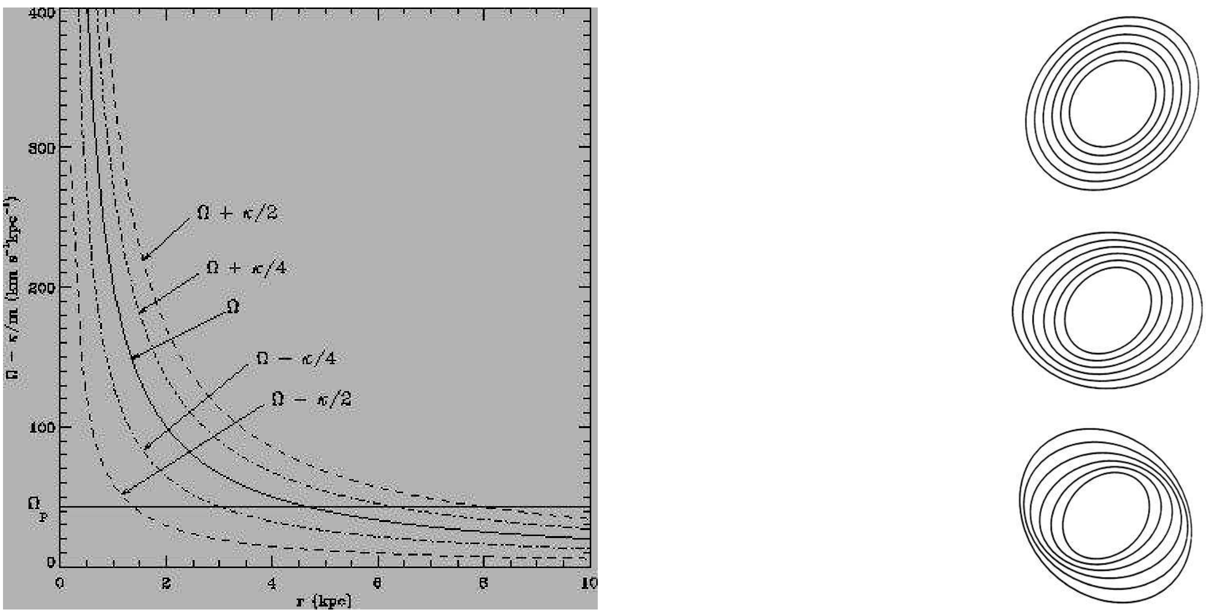

Figure 4.1: (a) The location of Inner Lindblad resonance (ILR) and the outer Lindblad resonance (OLR) can be graphically found by examining the plot of angular frequency $\dot{\phi}$ versus radius R. The curve of Ω implies corotation, since that is the frequency corresponding to the circular velocity. The ILR and OLR can be found where the horizontal line Ω_p =constant intersects the curves $\Omega - \kappa/m$ and $\Omega + \kappa/m$. (b)In a rotating frame, Lindblad resonance orbits appear as closed ellipses centred at $R = 0$. Since $\Omega - \kappa/2$ can be almost constant over a large range of R, it is easily seen how a two-armed pattern can form from these ellipses in a differentially rotating disk.

Spiral structure

We can apply epicycles in constructing kinematic spiral waves. For example, consider a ring of test particles on similar epicyclic orbits with their guiding centres at the same radius r_g. let the initial phases of the epicycles be such that at $t = 0$ the particles describe an oval. With time the guiding centres travel around the galaxy with angular velocity Ω, but the stars at the ends of the oval are being carried backward with respect to their guiding centres, so the form of the oval advances more slowly. The rate of precession or "pattern speed" of the oval is given by

$$\Omega_p = \Omega - \frac{\kappa}{2}.$$

By superposing ovals of different sizes, one can produce a variety of spiral patterns (see discussion in Sparke & Gallagher and in BT.) If $\Omega - \kappa/2$ were independent of radius, such patterns would persist indefinitely because all the superposed ovals would precess at the same speed. In fact, for a wide range of plausible disk galaxy models, $\Omega - \kappa/2$ is found to be fairly constant over a large range of radii (see S&G §5.4 for a plot for the Plummer potential; also BT Figure 6.10). Compared to material arms, density waves in our Galaxy would wind up six times less rapidly, yielding predicted pitch angles of about 1.4 deg. This is an improvement, but still not consistent with observations This simple-minded model has neglected the self-gravity of spiral structure – so it cannot be telling the whole story. We won't go into the mechanism of "swing amplification" here– read up on it in a good textbook if you are interested.

In general, for an m-armed spiral, the pattern speed is given by

$$\Omega_p = \Omega - \frac{\kappa}{m},$$

such that stars orbiting at radius r pass though an arm of an m-armed spiral with frequency $m\,[\Omega_p - \Omega(r)]$. Spiral pattern persist if $m\,|\Omega_p - \Omega(r)| < \kappa(r)$, i.e. in the region between $\Omega_p = \Omega \pm \kappa/m$, the *inner* and *outer Lindblad resonances*. Since κ is usually bounded below by Ω (Keplerian value) and above by 2Ω (solid-body-rotation value), the $m = 1$ (one-armed spiral) and $m = 2$ (two-armed spiral) disturbances may have no inner Lindblad resonance if Ω_p has a sufficiently large positive value (see Figure).