Quasistatic Equilibrium Models of Galaxy Formation and the Consequences of Stochasticity

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2015 September
In the Footsteps of Galaxies
Or How I Learned to Stop Worrying
and
Love the Central Limit Theorem

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In the Footsteps of Galaxies
A Malaise in Galaxy Evolution

- Incomplete knowledge limits us to oversimplified histories
- Toy models used as a substitute for physical understanding
- There are empirical constraints on average histories, but not real ones
- Dependent on assumed halo merger trees
- All sims/SAMs dependent on a lot of subgrid physics
My Starting Point

• Tinsley & Larson (1978), Efstathiou (2000): galaxies in steady-state between inflows, outflows, feedback

• Quasi-static equilibria imply evolving mass growth rates with $E[\Delta dM/dt] = 0$

• MCLT allows us to derive: $E[dM/dt]$, $E[M]$, $E[d\ln M/dt]$, and $\text{Sig}[d\ln M/dt]$
We do Everything in Ways Directly Related to SSFR

This is the/an SDSS correlation of SFR vs stellar mass:

Common interpretation: more massive galaxies make more stars
This correlation is also seen at high redshift

(Peng et al 2010)
How Does This Correlation Reflect Galaxy Evolution?

This is a much better SDSS view of Specific SFR vs stellar mass:

Intrinsic scatter ~0.4 dex in SSFR, relatively constant in M

SSFR vs M: relatively flat below log $M < 9.5$, anticorrelated at higher masses

Change SSFR to SFR per unit disk mass: a lot of the slope goes away

Thus expect late-time bulge formation $\rightarrow$ flatter SFMS at higher $z$,

(Salim et al 2007, Abramson et al 2014)
Oh, Look, We Were Right

Constant $\langle$SSFR$\rangle$ at low-mass continued to high $z$; break at higher mass

There is clearly something fundamental about this flat part of the SFMS!

(Whitaker et al 2014)
Let us begin: Assumption 1 — Steady-state

Given a sequence of stellar mass growths at interval $t$, $S_0, S_1, S_2, \ldots, S_{t+1}$, let us define $X_{t+1}$,*

$$X_{t+1} = S_{t+1} - S_t$$

In other words,

$$S_t = (S_t - S_{t-1}) + (S_{t-1} - S_{t-2}) + (S_{t-2} - S_{t-3}) + \cdots + S_0$$

$$S_t = \sum_{i=1}^{t} X_i$$

And remember that

$$M_{t+1} = \sum_{i=1}^{t} S_i$$

$$M_{t+1} = \sum_{i=1}^{t} \sum_{j=1}^{i} X_j$$

* Warning: astrophysics buried here.
Let us begin: Assumption 1 — Steady-state

So the SFMS is apparently just a correlation between $\sum_{i}^{t} X_i$ and $\sum_{i}^{t} \sum_{j}^{i} X_j$.

Can we work out how those two sums should be correlated?

$S$ is stationary, so $E[X] = 0$. But there is a variance $\sigma_{t}^{2}$:

$$\text{Var}[S_{t} - S_{t-1}] = \sigma_{t}^{2}$$

Believe it or not, we now have almost everything we need to compute a lot of the evolution of the cosmic ensembles of galaxies!

* Astrophysics buried here!
The Martingale Central Limit Theorem

If the stochastic differences, $X$, are i.r.v. centered on zero, then $S$ is called a “martingale,” and $X$ are “martingale differences.”

Why do you care about this?

Sums of sequences of such numbers obey central limit theorems.

If you have central limit theorems, you can compute probabilities!

(Strap yourself in for the ride now.)
The Martingale Central Limit Theorem

We need to compute the variance in $S_t$: 

$$\text{Var}[S_t] = E[S_t^2] - (E[S_t])^2$$

Given that $S$ is stationary, centered on $S_0 = 0$, $E[S_t] = 0$, and thus 

$$\text{Var}[S_t] = \sum_{i=1}^{t} X_i^2 = \sum_{i=1}^{t} \sigma_i^2$$

where $\sigma_i$ is the expected variance in the stochastic changes to $S$ at time $i$.

Let’s take an ensemble of $N$ object histories $S_{n,t}$, where $n \in \{1, 2, 3, \ldots, N\}$.

Each object, $n$, has a history, with different variances at every timestep, etc.

We therefore define an RMS stochastic fluctuation for $n$’s history up to $S_{n,t}$:

$$\bar{\sigma}_{n,t} = \left( \frac{1}{t} \sum_{i=1}^{t} \sigma_{n,i}^2 \right)^{1/2}$$
The Martingale Central Limit Theorem

Note these RMS stochastic fluctuations for each $n$ history up to time $t$,\[
\bar{\sigma}_{n,t} = \left( \frac{1}{t} \sum_{i=1}^{t} \sigma_{n,i}^2 \right)^{1/2}
\]
have all the physics.

The central limit theorem states that the distribution of $S_{n,t}$, normalized by these RMS fluctuations,\[
\frac{S_{n,t}}{t^{1/2}\bar{\sigma}_{n,t}} = \frac{1}{t^{1/2}\bar{\sigma}_{n,t}} \sum_{i=1}^{t} X_{n,i}
\]
is a Gaussian centered in zero with a standard deviation of unity:\[
\frac{S_{n,t}}{t^{1/2}\bar{\sigma}_{n,t}} \xrightarrow{d} N(0, 1)
\]
Imposing Nonnegativity

Stellar mass growth is almost always nonnegative.

Imposing $S \geq 0$ turns $S$ into a submartingale, and $S$, on average tends to go up. Every submartingale can be expressed as the sum of:

1. a martingale (yay!), and

2. a long-term drift term

The resulting limit for $S \geq 0$ is the nonnegative half of the Gaussian:

$$P\left[\frac{S_{n,t}}{t^{1/2} \sigma_{n,t}} < x\right] = \left(\frac{2}{\pi}\right)^{1/2} \int_0^x e^{-x^2/2} dx$$

WE NOW HAVE A PROBABILITY DISTRIBUTION.

Let us now skip doing the integrals and just write down the 1st and 2nd moments.
Markovian Expectation Values

So far we have derived a probability distribution for $S_t$ assuming the timesteps are independent of each other, and galaxies at time $t$ don’t care what they’ve done previously.

You get 1st and 2nd moments of $dP/dx$, plus the integral of the 1st moment:

$$E\left[\frac{S_t}{\sigma}\right] = \left(\frac{2}{\pi}\right)^{1/2} t^{1/2}$$

$$\text{Var}\left[\frac{S_t}{\sigma}\right] = \frac{1}{2} E\left[\frac{S_t}{\sigma}\right]^2$$

$$E\left[\frac{M_t}{\sigma}\right] = \left(\frac{2}{3}\right)\left(\frac{2}{\pi}\right)^{1/2} t^{3/2}$$
A Markovian Star-Forming Main Sequence

If galaxies grow in a sort of steady-state, with stochastic changes to their growth rates, and every stochastic change to a galaxy’s growth rate is independent of the other stochastic changes in its history, one gets this SFMS:

\[ E\left[\frac{S_t}{M_t}\right] = \left(\frac{3}{2t}\right) \]

\[ \text{Sig}\left[\frac{S_t}{M_t}\right] = \frac{1}{\sqrt{2}} E\left[\frac{S_t}{M_t}\right] \]

\[ \text{Sig}\left[\ln\frac{S_t}{M_t}\right] \approx \frac{1}{\sqrt{2}} \]

\[ \text{Sig}\left[\log\frac{S_t}{M_t}\right] \approx 0.3 \text{ dex} \]

But the bad news is that galaxies aren’t Markovian.
Covariant Stochasticity: Timesteps are Correlated

In reality, a galaxy’s history has long- and short-term correlations between stochastic changes to its growth:

\[ S_t = \sum_{i=1}^{t} \sum_{j=0}^{m} c_{i,i-j} X_{i-j} \]

There is an unknown, seemingly unconstrained set of covariances between stochastic changes in \( S \).

Guess what: sums of \( m \)-dependent random variables also obey limit theorems!
Covariant Stochasticity: Convergence in Distribution

Convergence of weighted sums of random variables with long-range dependence

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Abstract

Suppose that $f$ is a deterministic function, $\{\xi_n\}_{n \in \mathbb{Z}}$ is a sequence of random variables with long-range dependence and $B^H$ is a fractional Brownian motion (fBm) with index $H \in (\frac{1}{2}, 1)$. In this work, we provide sufficient conditions for the convergence

$$\frac{1}{m^H} \sum_{n=-\infty}^{\infty} f\left(\frac{n}{m}\right) \xi_n \to \int_{\mathbb{R}} f(u) \, dB^H(u)$$

in distribution, as $m \to \infty$. We also consider two examples. In contrast to the case when the $\xi_n$’s are i.i.d. with finite variance, the limit is not fBm if $f$ is the kernel of the Weierstrass–Mandelbrot process. If however, $f$ is the kernel function from the “moving average” representation of a fBm with index $H'$, then the limit is a fBm with index $H + H' - \frac{1}{2}$. © 2000 Published by Elsevier Science B.V.
Covariant Stochasticity: fractional Brownian motion

FRACTIONAL BROWNIAN MOTIONS, FRACTIONAL NOISES AND APPLICATIONS*

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1. Introduction. By “fractional Brownian motions” (fBm’s), we propose to designate a family of Gaussian random functions defined as follows: \( B(t) \) being ordinary Brownian motion, and \( H \) a parameter satisfying \( 0 < H < 1 \), fBm of exponent \( H \) is a moving average of \( dB(t) \), in which past increments of \( B(t) \) are weighted by the kernel \( (t - s)^{H-1/2} \). We believe fBm’s do provide useful models for a host of natural time series and wish therefore to present their curious properties to scientists, engineers and statisticians.

The basic feature of fBm’s is that the “span of interdependence” between their increments can be said to be infinite. By way of contrast, the study of random functions has been overwhelmingly devoted to sequences of independent random variables, to Markov processes, and to other random functions having the property that sufficiently distant samples of these functions are independent, or nearly so. Empirical studies of random chance phenomena often suggest, on the contrary, a strong interdependence between distant samples. One class of examples arose in economics. It is known that economic time series “typically” exhibit cycles of all orders of magnitude, the slowest cycles having periods of duration comparable to the total sample size. The sample spectra of such series show no sharp “pure period” but a spectral density with a sharp peak near frequencies close to the inverse of the sample size \([1], [4]\). Another class of examples arose in the study of fluctuations in solids. Many such fluctuations are called “1:1 noises,” because their sample spectral density takes the form \( \lambda^{1-2H} \), with \( \lambda \) the frequency, \( \frac{1}{2} < H < 1 \) and \( H \) frequently close to 1. Since, however, values of \( H \) far from 1 are also frequently observed, the term “1:1 noise” is inaccurate. It is also unwieldy. With some trepidation due to the availability of
Covariant Stochasticity: fractional Brownian motion

2. The definition of fractional Brownian motion. As usual, \( t \) designates time, \( -\infty < t < \infty \), and \( \omega \) designates the set of all the values of a random function. (This \( \omega \) belongs to a sample space \( \Omega \).) The ordinary Brownian motion, \( B(t, \omega) \), of Bachelier, Wiener and Lévy is a real random function with independent Gaussian increments such that \( B(t_2, \omega) - B(t_1, \omega) \) has mean zero and variance \( |t_2 - t_1| \), and such that \( B(t_2, \omega) - B(t_1, \omega) \) is independent of \( B(t_4, \omega) - B(t_3, \omega) \) if the intervals \( (t_1, t_2) \) and \( (t_3, t_4) \) do not overlap. The fact that the standard deviation of the increment \( B(t + T, \omega) - B(t, \omega) \), with \( T > 0 \), is equal to \( T^{1/2} \) is often referred to as the "\( T^{1/2} \) law."

**Definition 2.1.** Let \( H \) be such that \( 0 < H < 1 \), and let \( b_0 \) be an arbitrary real number. We call the following random function \( B_H(t, \omega) \), reduced fractional Brownian motion with parameter \( H \) and starting value \( b_0 \) at time 0. For \( t > 0 \), \( B_H(t, \omega) \) is defined by

\[
B_H(0, \omega) = b_0,

B_H(t, \omega) - B_H(0, \omega)
\]

\[
(2.1) = \frac{1}{\Gamma(H + \frac{1}{2})} \left\{ \int_{-\infty}^{0} [(t - s)^{H-1/2} - (-s)^{H-1/2}] dB(s, \omega)

+ \int_{0}^{t} (t - s)^{H-1/2} dB(s, \omega) \right\}
\]
fractional Brownian motion: the long and the short of it

We already derived what is effectively the Brownian case.

The fBm models are generalizations governed by the Hurst parameter: $0 \leq H \leq 1$.

When $H = 0.5$ timesteps are all independent (Brownian). When $H < 0.5$ there is antipersistence. When $H > 0.5$ there is persistence (positive feedback). Technically the bounds are not inclusive, because when $H = 1$ the integral only converges at $t = \infty$.

For such a case, it would be like the universe was a system with an ensemble of galaxies that never forgot their pasts.
Serving both the interests of the audience and the speaker, let us just jump to:

\[ E[S_t] = \bar{\sigma} \left( \frac{2}{\pi} \right)^{1/2} \left( \frac{t^H}{2H} \right) \]

\[ \text{Sig}[S_t] = H^{1/2} E[S_t] \]

\[ E[M_t] = \bar{\sigma} \left( \frac{2}{\pi} \right)^{1/2} \left[ \frac{t^{H+1}}{2(1 + H)H} \right] \]
Nonnegative fBm: Example Scale-Free Histories

![Graph showing scale-free histories with different Hurst exponents](image)
Nonnegative fBm: Example Scale-Free Growth Histories
Nonnegative fBm: The Star-Forming Main Sequence

The expectation values for $S_t$ and $M_t$, again, are both proportional to $\bar{\sigma}$.

Thus one obtains a generalized SFMS of:

$$E[S_t/M_t] = \frac{(H + 1)}{t}$$

$$\text{Sig}[S_t/M_t] = H^{1/2}E[S_t/M_t]$$

The amazing thing about this result is that the predicted scatter is independent of any long-term drift changes in expectations (such as when galaxy environments evolve to sufficiently modify long-term expectations of gas supply, etc).
Nonnegative fBm: The Star-Forming Main Sequence

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In other words, systematic changes in long-term expectations will not affect the relative scatter.

So how do these equations compare to real data?
Back to the Star-Forming Main Sequence

Lots of data from the literature for the flat part of the SFMS, selecting those samples deep enough to not be biased against passive galaxies.

These data look like a fracking mess. How would when even begin to test whether the predictions are correct?
Rethinking those Measurements

Turns out that different people measure different things, artificially inflating the apparent disagreement among datasets.
The difference between a mean and a median

Recall that

\[ E[S_t/M_t] = \frac{(H + 1)}{t} \]
\[ \text{Sig}[S_t/M_t] = H^{1/2}E[S_t/M_t] \]

This scatter translates directly to an offset between the mean and median SSFR.

Let’s fit \( A/t \) to the mean SSFRs and \( B/t \) to the medians and compute \( \log A/B \):
So galaxies are a bit like elephants

Here the violet solid line is the predicted locus for Median[SSFR] vs redshift.

The violet dashed line is the predicted locus for the Mean[SSFR] vs redshift.

(a) $H_0=74.3$ km/s/Mpc
$\Omega_M=0.2892$
$\Omega_b=1-\Omega_M$
$H=1$

The Flat Part of the SFMS
There’s no way this is an accident

We derived that the Median\([S/M]\) on the flat-part of the SFMS is identically \(2/t\).

The implication is that every published Median\([S/M]\) is therefore a cosmic clock.

IOW: \(2/t\) goes right through the medians, and \(2/t \times 1.57\) right through the means. To a few pct.
What about the expectation value for the scatter?

In SDSS Salim et al (2007) quote 0.4 dex intrinsic.
At high redshift Gonzalez et al (2014) quote ~0.5 dex.

**Very difficult to measure right; selection biases matter a lot.**

Do not measure for SF gals only! Samples must be cosmologically representative!

What about the expectation value for the scatter?

Peng et al (2010)
Let Us Breath And Quickly Take Stock

• The SFMS is emergent.

• The SFMS does not imply that more massive galaxies form stars at greater rates!

• Rather: in order for a galaxy of mass \( M \) to have formed by \( z \), it had to have formed stars more vigorously than lower mass galaxies.

• Correlation does not imply causation, except in this case, star-formation causes stellar mass.

• The set of SFHs implied by fBm is quite diverse (and infinite).

• Implied histories show activity on a range of timescales, such that, e.g., “quiescent” SFHs aren’t actually SFR=0, let alone forever.
Galaxy Evolution at Early Times

The lack of dependence of SSFR on M at early times implies we have a fully formed model of galaxy ensembles at those epochs.

Except that we derived: \[ E[M_t] = \bar{\sigma} t^2 / (2\sqrt{2\pi}). \]

Up until now, we have treated \( \bar{\sigma} \) as a nuisance, as something we can ignore.

But \( \bar{\sigma} \) normalizes the SFRs and stellar masses, and is thus critical for computing stellar mass functions over time!

\textit{Can we calculate} \( \bar{\sigma} \) \textit{a priori?}
A Characteristic Stochastic Fluctuation Amplitude

Let us start with

\[ E \left[ \frac{dM}{dt} \right] = \frac{\sigma}{\sqrt{2\pi}} t \]

Let us then take the first derivative (investigate ensembles for which the RMS fluctuation is roughly constant over some time interval):

\[ \frac{d}{dt} E \left[ \frac{dM}{dt} \right] = \frac{\sigma}{\sqrt{2\pi}} \]

\[ E \left[ \frac{d^2 M}{dt^2} \right] = \frac{\sigma}{\sqrt{2\pi}} \]

Let us simplify \( \frac{dM}{dt} \) as the rate of accretion of baryons, converted to stars with some fraction \( \epsilon \), where \( v_b \) is the infall velocity and \( \rho_b \) is the ambient density:

\[ \frac{dM}{dt} = \epsilon \rho_b v_b \]
A Characteristic Stochastic Fluctuation Amplitude

We’ll use a simple top-hat approximation, and other assumptions about the density of the ambient medium being relatively constant over a short enough timescale at the start of the stochastic process $S$, so that:

$$\frac{d^2 M}{dt^2} = \epsilon \rho_b \frac{dv_b}{dt}$$

$$= \epsilon \rho_b \frac{GM_h}{R_h^2}$$

which eventually will look like

$$\frac{d^2 M}{dt^2} = \epsilon f_b \left(\frac{4\pi 178}{3}\right)^{2/3} GM_h^{1/3} \rho^{5/3}$$

Using characteristic halo mass at the onset of star-formation, and the matter density at that epoch, one then has a characteristic $d^2M/dt^2$, and thus a characteristic $\sigma^*$. 
A Characteristic Stochastic Fluctuation Amplitude

Popular halo mass functions for $z \sim 10$ have characteristic $M_h \sim 6 \times 10^9 M_\odot$ (e.g. Warren et al 2006, Tinker et al 2008).

Let us adopt a rate of conversion of baryons to stars of 2%, and baryon fraction $f_b = 0.15$.

This number is what goes in front of, e.g., $E[M_t] = \bar{\sigma} t^2 / (2\sqrt{2\pi})$:

$$\bar{\sigma}^* \approx \left( \frac{\epsilon}{0.02} \right) \left( \frac{f_b}{0.15} \right) \left( \frac{1 + z}{1 + 10} \right)^5 \left( \frac{M_h}{6 \times 10^9 M_\odot} \right)^{1/3} \times \left( 1.0 \times 10^{-7} M_\odot/\text{yr}^2 \right)$$
High-z Stellar Mass Functions and Madau Diagram

Let’s use this characteristic SF acceleration and define a spectrum of values for galaxy seeds, and...

(SFRD from Madau & Dickinson 2014; mass functions from various.)
Implications for the Scatter in SSFR at Fixed Mass

The intrinsic scatter in SSFR means “quiescent” often $\neq$ dead

(Data from Tomczak et al 2014)

Over long baselines in $z$, galaxies below the median will move above, and vice versa.
A Quick Not-Quite-Right Toy to Get to Low-z

The implied amount of DM matches that of the universe at $z \sim 3$, when the observed SFMS starts really being modified at the high mass end.

I haven’t had time yet to write the proper lines of code to properly extinguish those galaxies, but:

Pretty obvious which galaxies make it into the groups, eh?
Some of the Relevant Stopping Points

- The “Star-Forming Main Sequence” is emergent, and a natural consequence of stellar mass growth as a (non-Markovian) stochastic process.
- Derive $E[(dM/dt)/M] = 2/t$, accurately matching SSFRs over $0 < z < 10$ with error of 3%.
- Observed intrinsic scatter in SSFR at fixed mass falls right out.
- Stellar mass functions and Madau diagram $3 \lesssim z \lesssim 10$.
- Infinite set of possible SFHs, including those of local group dwarf gals, MW.
- Retrodict quiescent galaxy fractions along flat part of SFMS.
- Strongly limits how well one can link specific progenitors with specific descendents.
- Must trace full ensembles over cosmic time, but we now have math to help us!
- This framework is not yet complete — must incorporate a little more physics to get long-term evolution of massive galaxies (merging? gas depletion? AGN?)
- Very simply explains rising SFHs for early galaxies.
- Leads to characteristic mass growth as $t^2$; naturally produces characteristic stellar mass scales at all epochs.