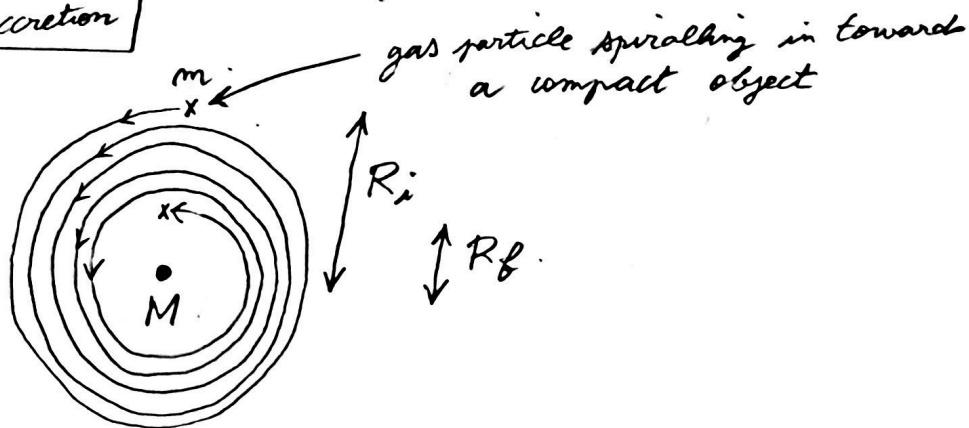


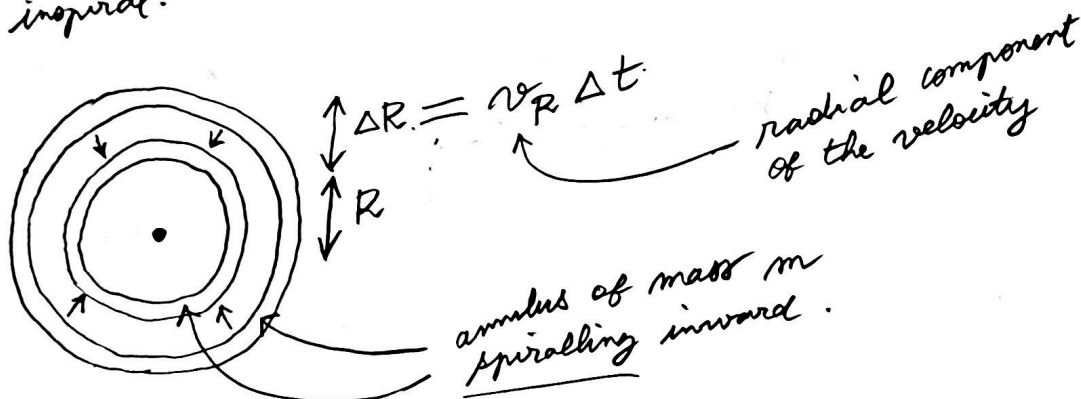
Accretion



$$\Delta E_{\text{orb}} \approx -\frac{GMm}{R_f} + \frac{GMm}{R_i} < 0 \Rightarrow$$

change of orbital energy caused by the inspiral.

"Accretion" energy
 $E_{\text{acc}} = -\Delta E_{\text{orb}}$
 released that can be radiated.



$$\Delta E_{\text{acc}} \approx GMm \left(\frac{1}{R} - \frac{1}{R+\Delta R} \right) \approx \frac{GM}{R} \cdot \frac{m \Delta R}{R}$$

assuming $\Delta R \ll R$.

$$\frac{\Delta E_{\text{acc}}}{\Delta t} = \frac{GM}{R} \left(\frac{m}{\tau_{\text{acc}}} \right)$$

accretion timescale
 $\tau_{\text{acc}} = R/v_R$

|||
 L
 (accretion luminosity)

|||
 $\dot{M}$
 (accretion rate)

So,

$$L = \frac{GM}{R} \dot{M}$$

$$L = \frac{GM}{R} \dot{M} = \frac{GM}{R_g} \frac{R_g}{R} \dot{M} = \eta \dot{M} c^2$$

$R_g \equiv \frac{GM}{c^2}$
 (gravitational radius)

$\eta \equiv \frac{R_g}{R} < 1$
 (radiative efficiency)

Accretion can release a fraction η of the gas rest mass energy. The radiative efficiency increases as R gets closer to R_g .

Innermost radius for accretion: R_{in}

- Neutron stars, white dwarfs: physical radius of the object ($R_{in} = R_{NS}, R_{WD}$).
- Black holes: radius of the innermost stable circular orbit ($R_{in} = 6 R_g$ for Schwarzschild, $R_{in} = 1 R_g - 9 R_g$ for Kerr metric)

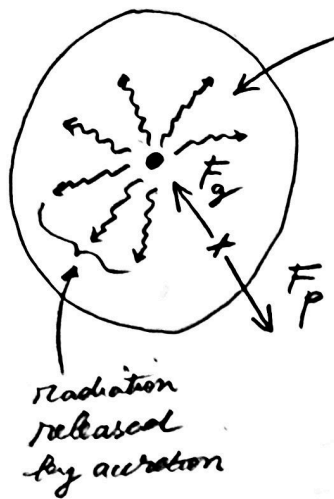
$\nearrow$ co-rotating case $\nwarrow$ counter-rotating case

(i.e. accreting gas with the same direction of rotation as the black hole) -

[$\eta = 1$ in this case with our Newtonian treatment, but this is prevented in general relativity, with the maximum being $\eta \approx 42\%$.]

$\uparrow$
 see lecture
 on general-
 relativistic effects
 for a detailed derivation
 of the radiative efficiency.

Eddington luminosity



accreting spherical gas cloud.

gravitational force.

$$F_g > F_p$$

radiation pressure force.

for accretion to be possible, i.e. the gas must be able to move inward.

$$F_p = \frac{\Delta p}{\Delta t}$$

momentum that gets deposited to the gas shell during Δt : $\Delta p = \frac{\Delta E}{c} \times \frac{\sigma_T}{4\pi R^2}$

$$\Delta E = L \Delta t$$

($\sigma_T \approx 10^{-24} \text{ cm}^2$)
Thomson cross section.

$$F_p = \frac{L \sigma_T}{4\pi R^2 c}$$

momentum passing through the shell during Δt .

$$p = \frac{E}{c}$$

for photons.
(E is the outgoing radiation energy generated by accretion)

fraction of the outgoing momentum that gets deposited to electrons through Thomson scattering (this momentum also gets transferred to protons via the electrostatic force).

$$F_g = \frac{G M m_p}{R^2}$$

gravitational force from the central object on protons.

$$\text{So: } \frac{G M m_p}{R^2} > \frac{L \sigma_T}{4\pi R^2 c}$$

$$L < L_{\text{Edd}} \equiv \frac{4\pi G M m_p c}{\sigma_T}$$

Eddington luminosity limit

$$\text{Numerically: } L_{\text{Edd}} \approx 10^{38} \text{ erg/s } \left(\frac{M}{M_\odot} \right)$$

For blackbody radiation at $L = L_{\text{Edd}}$ from $R = R_{\text{in}}$:

$$T = \left(\frac{L_{\text{Edd}}}{4\pi R_{\text{in}}^2 \sigma} \right)^{1/4} \approx 10^8 \text{ K} \left(\frac{R_{\text{in}}}{R_g} \right)^{1/2} \left(\frac{M}{M_0} \right)^{-1/4}$$

weak dependence on the mass.

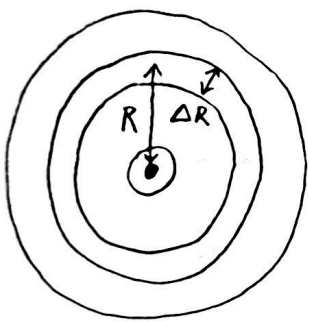
$$\left(\frac{10^{38}}{10 \times (10^5)^2 \times 10^{-5}} \right)^{1/4} = 10^{\frac{33}{4}} \approx 10^8$$

$\uparrow$
 $R_g \approx 1 \text{ km}$
for $M = M_0$

$$E = kT = 10^{-16} \times 10^8 \times \underbrace{10^{11}}_{\substack{\uparrow \\ \text{eV/erg}}} = 1 \text{ keV} \Rightarrow$$

Emission in the X-ray band.

Emitted spectrum from an accretion disc



R_{out}
 R_{in}

$$\sigma T^4(R) = \frac{\left| \Delta \left(\frac{GM\dot{M}}{R} \right) \right|}{2 \times 2\pi R \Delta R} \approx \frac{GM\dot{M}}{R^3}$$

$\uparrow$
blackbody flux emitted from disc annulus at radius R .

$\nwarrow$
emission from two sides

$$L\nu \propto \int_{R_{\text{in}}}^{R_{\text{out}}} I_\nu(T(R)) \times 2\pi R dR \propto \int_{R_{\text{in}}}^{R_{\text{out}}} \frac{\nu^3 R dR}{e^{\frac{h\nu}{kT(R)}} - 1}$$

$\nwarrow$
blackbody spectrum.

$$B_\nu(T(R)) \propto \frac{\nu^3}{e^{\frac{h\nu}{kT(R)}} - 1}$$

$\uparrow$
luminosity per bin of frequency
 $L\nu \equiv \frac{dL}{d\nu}$

Three regimes:

— Low frequencies $\left(\nu \ll \frac{kT(R_{\text{max}})}{h} \right)$: $\nwarrow$ minimal disc temperature

$$e^{\frac{h\nu}{kT(R)}} \approx 1 + \frac{h\nu}{kT(R)}$$

$\uparrow$
 $h\nu \ll kT(R_{\text{max}}) < kT(R)$

$$L_\nu \propto \int_{R_{in}}^{R_{out}} \frac{\nu^3 R dR}{h\nu/kT(R)} \propto \nu^2.$$

— High frequencies ($\nu \gg \frac{kT(R_{in})}{h}$)

$$e^{h\nu/kT(R)} - 1 \approx e^{h\nu/kT(R)}.$$

$$h\nu \gg kT(R_{in}) > kT(R)$$

$$L_\nu \propto \int_{R_{in}}^{R_{out}} \frac{\nu^3 R dR}{e^{h\nu/kT(R)}} \approx e^{-h\nu/kT(R_{in})}$$

↑
dominated by the
hottest part of the
disc at $R \approx R_{in}$

— Intermediate frequencies ($\frac{kT(R_{out})}{h} \ll \nu \ll \frac{kT(R_{in})}{h}$)

$$L_\nu \propto \int_{R_{in}}^{R_{out}} \frac{\nu^3 R dR}{e^{h\nu/kT(R)} - 1} \approx \int_{x(R_{in})}^{x(R_{out})} \frac{\nu^3 \nu^{-4/3} x^{4/3} \nu^{-4/3} x^{1/3} dx}{e^x - 1}$$

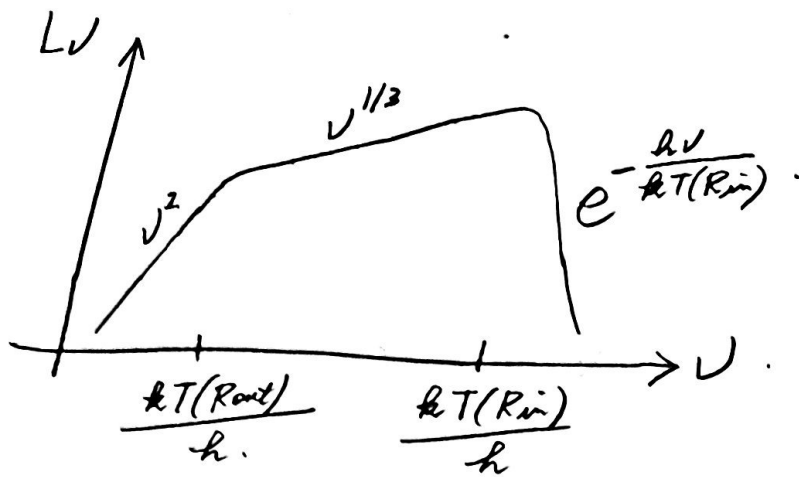
change of
variable:

$$\left\{ \begin{array}{l} \bullet x = \frac{h\nu}{kT(R)} = \frac{h\nu}{kT_{in}} \left(\frac{R}{R_{in}} \right)^{3/4} \Rightarrow R \propto \nu^{-4/3} x^{4/3} \\ \quad \quad \quad \uparrow \\ \quad \quad \quad T \propto R^{-3} \quad \quad \quad T(R_{in}) \\ \bullet dR \propto \nu^{-4/3} x^{1/3} dx \end{array} \right.$$

$$S_0: L_\nu \propto \nu^{1/3} \int_{x(R_{in})}^{x(R_{out})} \frac{x^{5/3}}{e^x - 1} dx \approx \nu^{1/3} \int_0^{+\infty} \frac{x^{5/3}}{e^x - 1} dx \propto \nu^{1/3}.$$

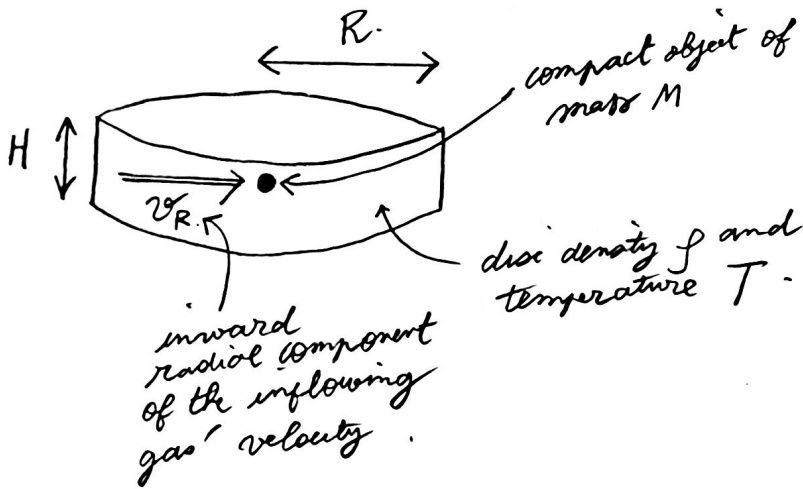
$$\uparrow \\ 3 - \frac{8}{3} = \frac{1}{3}$$

$$\left\{ \begin{array}{l} \bullet x(R_{out}) = \frac{h\nu}{kT(R_{out})} \gg 1 \\ \bullet x(R_{in}) = \frac{h\nu}{kT(R_{in})} \ll 1 \end{array} \right.$$



Multi-colour blackbody spectrum from accretion disc.

Accretion disc structure



("one-zone" model for an accretion disc)

The disc emits a luminosity L and evolves on a timescale $t_d = \frac{R}{v_R}$ determined by the inflow of gas. Given L , t_d , M , and R (observed), we will calculate H , ρ , and T (disc geometry and internal properties).

Three equations needed.

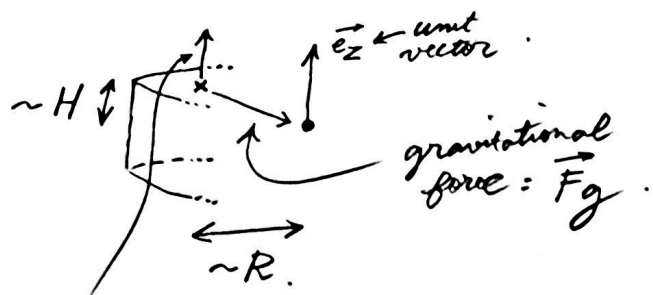
(1) Mass conservation :

$$\dot{M} = \underbrace{2\pi R H}_{\text{lateral disc surface}} \underbrace{\rho v_R}_{\text{mass flux}}$$

mass inflow rate related to the luminosity via

$$L = \frac{GM\dot{M}}{R}$$

(2) Vertical hydrostatic equilibrium



pressure force $\vec{F}_p$

$$\vec{F}_p \cdot \vec{e}_z + \vec{F}_g \cdot \vec{e}_z = 0 \quad \left(\text{hydrostatic equilibrium in the vertical } \vec{e}_z \text{-direction} \right)$$

$$\frac{-\vec{\nabla} P}{\rho} \cdot \vec{e}_z \approx \frac{P}{\rho H} \vec{e}_z \quad - \frac{GMH}{(R^2 + H^2)^{3/2}} \approx -GMH/R^3$$

assuming $H \ll R$.

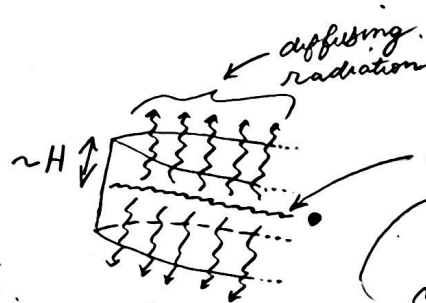
$$\frac{P}{\rho H} \approx \frac{GMH}{R^3}$$

$$c_s \approx \left(\frac{P}{\rho} \right)^{1/2} \rightarrow \text{ss} \quad v_K = \left(\frac{GM}{R} \right)^{1/2} \quad \left(\text{circular Keplerian velocity} \right)$$

(sound speed) $\frac{c_s^2}{H}$ $v_K^2 H/R^2$

$$\text{So: } \frac{H}{R} \approx \frac{c_s}{v_K}$$

(3) Radiative transport



$$L = \pi R^2 \times \underbrace{c_{\text{rad}}}_{aT^4 \text{ (radiation energy density)}} \times \underbrace{v_{\text{diff}}}_{\text{see lecture on supernovae}}$$

$\frac{c}{\tau} \leftarrow \tau \approx \kappa \rho H$

(Radiation produced near the disk's midplane and transported vertically to the surface by diffusion.)

$$\text{So: } L = \pi R^2 \frac{aT^4 c}{\kappa \rho H}$$

Three equations.
(order of magnitude level).

$$\begin{cases} \dot{M} \approx R H \rho v_R \Rightarrow \frac{L}{GM} \approx \frac{H \rho R}{\epsilon \dot{M}} \quad (1) \\ \frac{H}{R} \approx \frac{c_s}{v_K} \Rightarrow \left(\frac{H}{R} \right)^2 \approx \frac{aT^4 R}{\rho GM} \quad (2) \\ L \approx R^2 \frac{aT^4 c}{\kappa \rho H} \quad (3) \end{cases}$$

$\begin{cases} v_R = R/\tau_d \\ L = G\dot{M}/R \end{cases}$

$c_s \approx (aT^4/\rho)^{1/2}$ assuming that radiation pressure dominates (correct at high temperatures).

Now solving (1), (2), (3) for H, ρ, T .

$$(2) \Rightarrow \left(\frac{H}{R}\right)^2 \approx \frac{a T^4 R}{\rho G M} \approx \frac{L K H}{G M c R} \Rightarrow \frac{H}{R} \approx \frac{L K}{G M c} \approx \frac{L}{L_{\text{Edd}}}$$

$$\frac{a T^4}{\rho} \approx \frac{L K H}{c R^2} \quad (3)$$

So: $\boxed{\frac{H}{R} \approx \frac{L}{L_{\text{Edd}}}}$

("thick" disc for a luminosity close to the Eddington limit)

$$\begin{aligned} L_{\text{Edd}} &\approx \frac{G M m_p c}{\sigma_T} \\ \kappa &= \sigma_T / m_p \end{aligned}$$

assuming the opacity is due to electron scattering (correct at high temperatures)
($\kappa = 0.34 \text{ cm}^2/\text{g}$)

$$(1) \Rightarrow \rho \approx \frac{L t_d}{G M H R} \approx \frac{L_{\text{Edd}} t_d}{G M R^2} \approx \frac{c t_d}{\kappa R^2}$$

$$\frac{H}{R} \approx L / L_{\text{Edd}} \quad L_{\text{Edd}} \approx \frac{G M c}{\kappa}$$

$$(3) \Rightarrow T \approx \left(\frac{L K \rho H}{a R^2 c} \right)^{1/4} \approx \left(\frac{L t_d}{a R^3} \right)^{1/4} \left(\frac{L}{L_{\text{Edd}}} \right)^{1/4}$$

$$\tau = K \rho H \approx \frac{c t_d}{R^2} H = \frac{c t_d}{R} \frac{L}{L_{\text{Edd}}}$$

Accretion disc from a tidal disruption event:
(disruption of a star by a supermassive black hole)

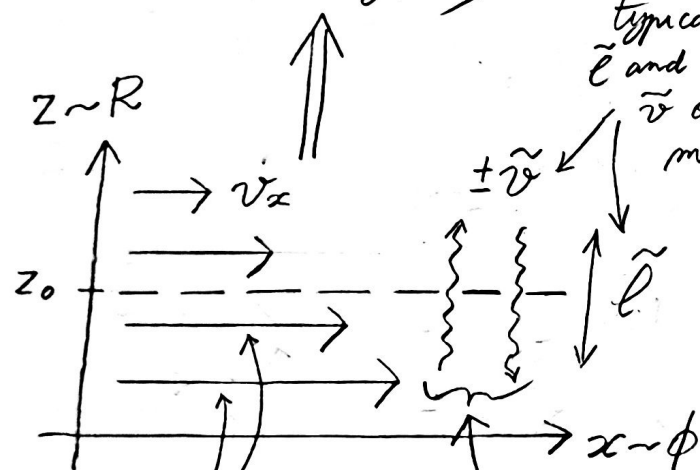
$$\begin{cases} t_d \approx 1 \text{ yr.} \\ L / L_{\text{Edd}} \approx 0.01 \\ R \approx 1 \text{ AU} \approx 10^{13} \text{ cm.} \\ M \approx 10^6 M_{\odot} \end{cases}$$

$$\begin{cases} \frac{H}{R} \approx 0.01 \equiv \frac{H_{\text{TDE}}}{R_{\text{TDE}}} \\ \rho \approx 2 \times 10^{-8} \text{ g/cm}^3 \equiv \rho_{\text{TDE}} \\ T \approx 4 \times 10^5 \text{ K} \equiv T_{\text{TDE}} \end{cases}$$

Need and origin of disc viscosity.

Viscosity is needed for the gas inside the accretion disc to flow inward while converting its orbital energy into heat. It is caused by a transport of momentum between fluid elements moving at different velocities around the compact object. This transport results from the chaotic motion of particles, for example due to thermal motion or turbulence.

Towards large radii
(moving away from the compact object).



« shear » flow velocity (v_x decreasing with z due to weaker gravity).

particles moving across the shear flow due to chaotic motion, conserving their x -momentum over the distance $\tilde{l}$.

typical scale $\tilde{l}$ and velocity $\tilde{v}$ of chaotic motion.

definition of viscosity ν .

$$\sigma_{xz} \equiv -\rho \nu \frac{dv_x}{dz}$$

« stress » $\equiv$ flux of x -momentum towards increasing z across the surface at $z = z_0$, also the force per unit area from the fluid element below this surface to the one above it.

$$\sigma_{xz} = \underbrace{\tilde{\rho} \tilde{v}}_{\substack{\text{mass flux} \\ \text{due to chaotic motion}}} \times \left[\underbrace{v_x(z_0 - \frac{\tilde{l}}{2}) - v_x(z_0 + \frac{\tilde{l}}{2})}_{\substack{\text{momentum per unit mass} \\ \text{of particles moving towards} \\ \text{increasing } z. (\frac{z}{R})}} \right] \approx -\rho \tilde{v} \tilde{l} \left(\frac{dv_x}{dz} \right)_{z=z_0}$$

same for decreasing z ($\frac{z}{R}$).
 $z \ll z_0$

$$\text{So: } \boxed{\nu = \tilde{l} \tilde{v}}$$

$\sigma_{xz} > 0 \Rightarrow x$ -momentum is transported towards larger z .

For an accretion disc, this implies that ϕ -momentum ($\equiv$ angular momentum) is transported towards larger radii R . The gas losing angular momentum spirals inwards, as needed to accrete onto the compact object.

The value of the viscosity (in cm^2/s) must satisfy:

$$t_{\text{visc}} \approx t_d$$

viscous timescale $\uparrow$ timescale over which the disc is observed to evolve $t_d \approx \frac{R}{v_R}$.

$$t_{\text{visc}} \approx \frac{R^2}{\nu}$$

from dimensional analysis $\uparrow$ radius of the disc.

For a tidal disruption event:
(above example)

$$(p_{\text{TDE}}, T_{\text{TDE}}, H_{\text{TDE}}) \approx \left(\frac{R^2}{t_d} \right)_{\text{TDE}} \approx \sqrt{v_{\text{TDE}}} \quad \text{III}$$

$$\approx 3 \times 10^{18} \text{ cm}^2/\text{s}.$$

disc properties $\uparrow$

$\begin{cases} R \approx 1 \text{ AU} \\ t_d \approx 1 \text{ yr} \end{cases}$

We will use this condition to identify a viable physical origin for the viscosity of an accretion disc.

Let's consider first "molecular" viscosity, for which momentum transport is due to thermal motion of particles. It is given by

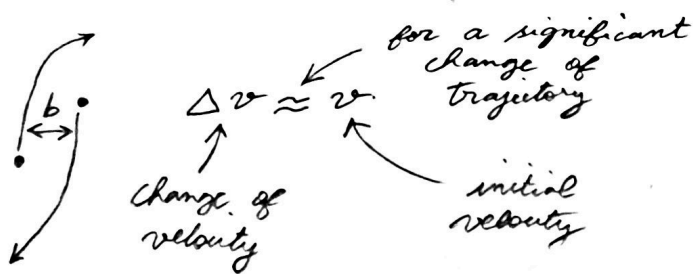
$$\nu_{\text{mol}} = \tilde{\ell} \tilde{v} \approx \lambda c_s.$$

$\begin{cases} \tilde{\ell} = \lambda \leftarrow \text{mean free path for Coulomb interactions} \\ \tilde{v} = c_s \leftarrow \text{sound speed} \end{cases}$

$$\lambda = \frac{1}{\sigma_c n} = \frac{m_p}{\rho \pi b^2}.$$

$\begin{cases} n = \rho/m_p \text{ (number density)} \\ \sigma_c = \pi b^2 \text{ (cross-section for Coulomb interactions)} \end{cases}$

$\uparrow$ typical separation between interacting particles.



$$\Delta v \approx \frac{a_c}{\frac{e^2}{b^2 m_p}} \Delta t \approx \frac{e^2}{m_p b v} \Rightarrow b \approx \frac{e^2}{m_p v^2} \approx \frac{e^2}{m_p c_s^2}$$

$\frac{e^2}{b^2 m_p}$ (Coulomb acceleration experienced by protons)
 $\Delta t \approx \frac{b}{v}$ (duration of the interaction)
 $v \approx c_s$ (thermal motion)

$$\lambda \approx \frac{m_p}{\rho \pi} \left(\frac{m_p c_s^2}{e^2} \right)^2 \approx \frac{m_p^2}{\rho} \left(\frac{c_s}{e} \right)^4$$

$$So: \nu_{mol} \approx \lambda c_s \approx \frac{m_p^2}{\rho} \frac{c_s^5}{e^4}$$

needed for the accretion disc to evolve on the observed timescale t_d .

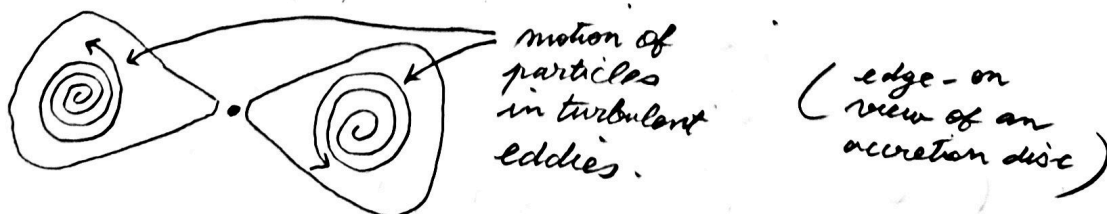
$$\nu_{mol}(\rho_{TDE}, T_{TDE}) \approx 10^{12} \text{ cm}^2/\text{s} \ll \nu_{TDE} \approx 3 \times 10^{18} \text{ cm}^2/\text{s}$$

$$\begin{cases} \rho_{TDE} \approx 2 \times 10^{-8} \text{ g/cm}^3 \\ T_{TDE} \approx 4 \times 10^5 \text{ K} \end{cases} \Rightarrow c_s = \left(\frac{\alpha T^4}{\rho} \right)^{1/2} \approx 1000 \text{ km/s}$$

$(e \approx 5 \times 10^{-10} \text{ statC})$
 c.g.s unit for charge.
 radiation pressure.

Molecular viscosity is not a viable physical origin for the viscosity of accretion discs.

Another possible source of viscosity is turbulence, for which momentum is transported through the formation of turbulent « eddies ».



Turbulent viscosity is given by:

$$\nu_{\text{turb}} \approx \tilde{v} \tilde{\ell} \lesssim c_s H$$

$$\left\{ \begin{array}{l} \cdot \tilde{v} \lesssim c_s \quad \left(\begin{array}{l} \text{subsonic motion to avoid} \\ \text{shocks that would prevent} \\ \text{the formation of eddies} \end{array} \right) \\ \cdot \tilde{\ell} \lesssim H \quad \left(\begin{array}{l} \text{the size of turbulent eddies cannot} \\ \text{exceed the width of the disc} \end{array} \right) \end{array} \right.$$

$$\nu_{\text{turb}}(p_{\text{TDE}}, T_{\text{TDE}}, H_{\text{TDE}}) \lesssim c_s H \approx 10^{19} \text{ cm}^2/\text{s} \approx 10 \nu_{\text{TDE}}.$$

$$\left\{ \begin{array}{l} \cdot c_s \approx 1000 \text{ km/s.} \\ \cdot H_{\text{TDE}} \approx 0.01 \underbrace{R_{\text{TDE}}}_{\substack{\text{ss} \\ 10^{13} \text{ cm}}} \approx 10^{11} \text{ cm.} \end{array} \right.$$

Turbulence is a viable source of viscosity for accretion discs, which is usually parametrized as $\nu = \alpha c_s H$ with $\alpha \approx 0.1$. The origin of turbulence is likely related to magnetic fields.