

Formation of compact objects

White dwarfs, neutron stars, black holes.

"Compactness" parameter = $\mathcal{C} = \frac{GM}{Rc^2} = \frac{R_g}{R}$

Annotations:
 G : mass of the object
 M : mass of the object
 R_g : gravitational radius (half of the Schwarzschild radius)
 R : radius of the object

General relativity becomes more important at increasing $\mathcal{C}$.

$\mathcal{C} \gtrsim 10^{-4}$ for a compact object.

Sun: $\left. \begin{matrix} M = M_\odot \\ R = R_\odot \end{matrix} \right\} \Rightarrow \mathcal{C} \approx 10^{-6} \text{ (X) (not a compact object)}$

Calculation:
 $R_g = \frac{GM}{c^2} \approx \frac{7 \times 10^{-8} \times 2 \times 10^{33}}{(3 \times 10^{10})^2} \approx \frac{10^{26}}{10^{21}} = 10^5 \text{ cm} \approx 10^{-6} R$
 (c.g.s units) $R = 7 \times 10^{10} \text{ cm}$

White dwarf: $\left. \begin{matrix} M \approx M_\odot \\ R \approx 10^{-2} R_\odot \end{matrix} \right\} \Rightarrow \mathcal{C} \approx 10^{-4} \text{ (V) (compact object)}$

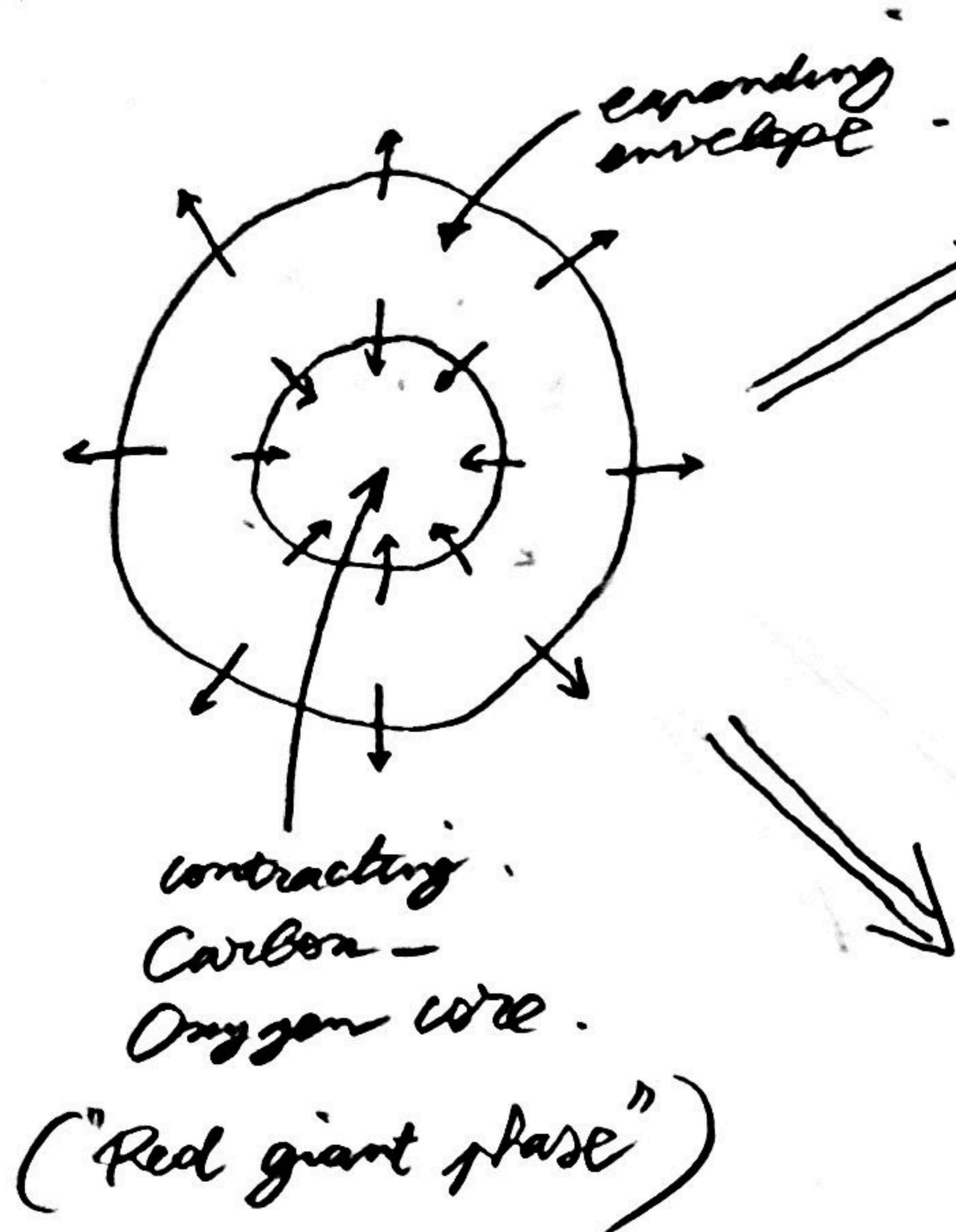
Neutron star: $\left. \begin{matrix} M \approx M_\odot \\ R \approx 10 \text{ km} \end{matrix} \right\} \Rightarrow \mathcal{C} \approx 10^{-1} \text{ (V)}$

$10 \text{ km} \approx 10^{-5} R_\odot$

Black hole: $\left. \begin{matrix} M \rightarrow \text{any} \\ "R" \approx R_g \end{matrix} \right\} \Rightarrow \mathcal{C} \approx 1 \text{ (V)}$

different types: {

- "stellar-mass" ($M \approx 1 - 100 M_\odot$)
- intermediate-mass ($M \approx 100 - 10^6 M_\odot$)
- supermassive ($M \approx 10^6 - 10^{11} M_\odot$)



Degenerate electron pressure takes over before carbon burning starts
($M_* \lesssim 8 M_\odot$)

$\Rightarrow$ white dwarf.

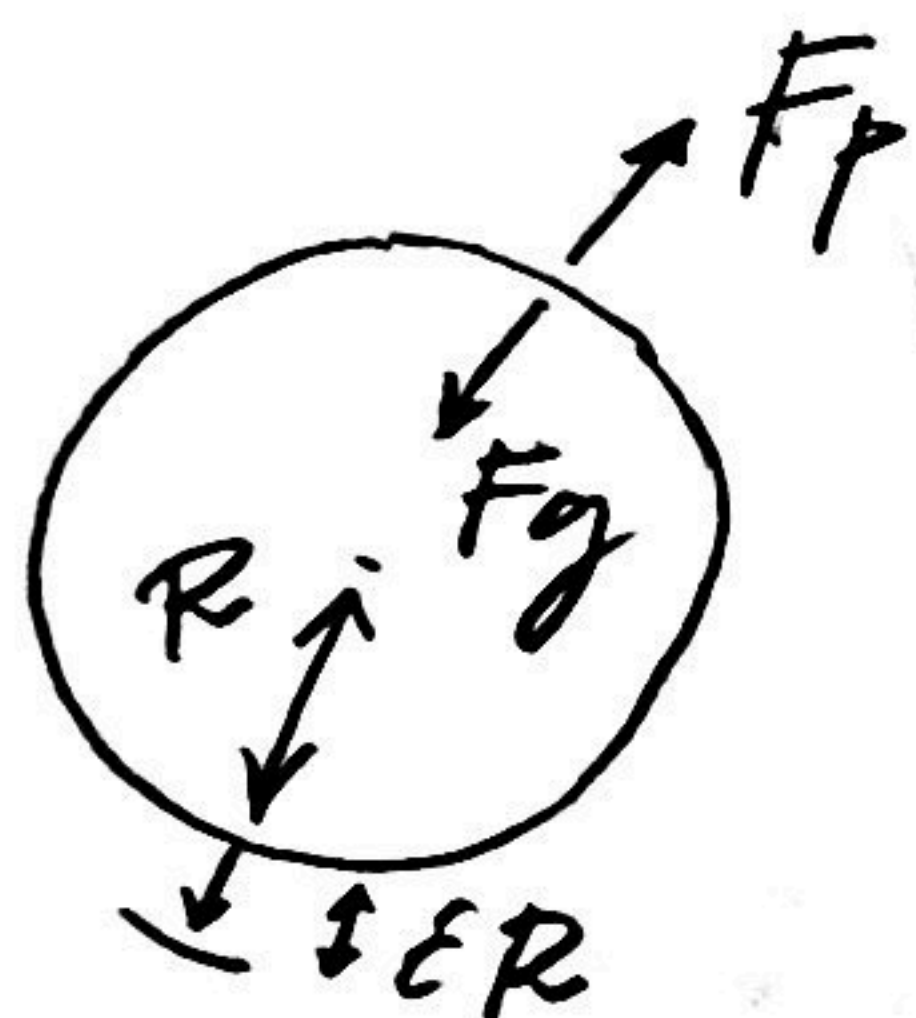
Burning continues until the formation of an iron core, which collapses.

($M_* \gtrsim 8 M_\odot$)

$\Rightarrow$ neutron star or (stellar-mass) black hole.

Because the fusion of iron cannot release energy to support gravity.

Dynamical stability



$$F_p = \frac{1}{\rho} \frac{dP}{dR} = F_g = \frac{GM}{R^2}$$

Hydrostatic equilibrium

$$P = K \rho^\gamma \quad \leftarrow \text{"adiabatic exponent"}$$

Small perturbation to the radius =

$$R' = R + \epsilon R = R(1 + \epsilon) \quad \leftarrow \epsilon \ll 1$$

$$F_p' \approx \frac{1}{\rho'} \frac{P'}{R'} \approx \frac{K(\rho')^{\gamma-1}}{R'} \approx \left(\frac{K \rho^{\gamma-1}}{R} \right) (1+\epsilon)^{-1} (1+\epsilon)^{3(1-\gamma)}$$

$\uparrow$ $\approx F_p$

$$\begin{cases} \bullet R' = R(1+\epsilon) \end{cases}$$

$$\begin{cases} \bullet \rho' \approx \frac{M}{(R')^3} \approx \frac{\rho}{(1+\epsilon)^3} \end{cases}$$

$\uparrow$ $\rho \approx M/R^3$

$$F_g' \approx \frac{GM}{(R')^2} \approx F_g (1+\epsilon)^{-2}$$

$$\text{so: } \frac{F_p'}{F_g'} \approx \frac{F_p}{F_g} (1+\epsilon)^{-1} (1+\epsilon)^{3(1-\gamma)} (1+\epsilon)^2 \approx (1-\epsilon)(1+3(1-\gamma)\epsilon)(1+2\epsilon)$$

$$\begin{cases} \bullet F_p \approx F_g \\ \bullet (1+\epsilon)^x \approx 1+x\epsilon \end{cases}$$

$$\approx 1 - \epsilon + 3(1-\delta)\epsilon + 2\epsilon = 1 + \epsilon(4-3\delta).$$

first order
in $\epsilon \ll 1$

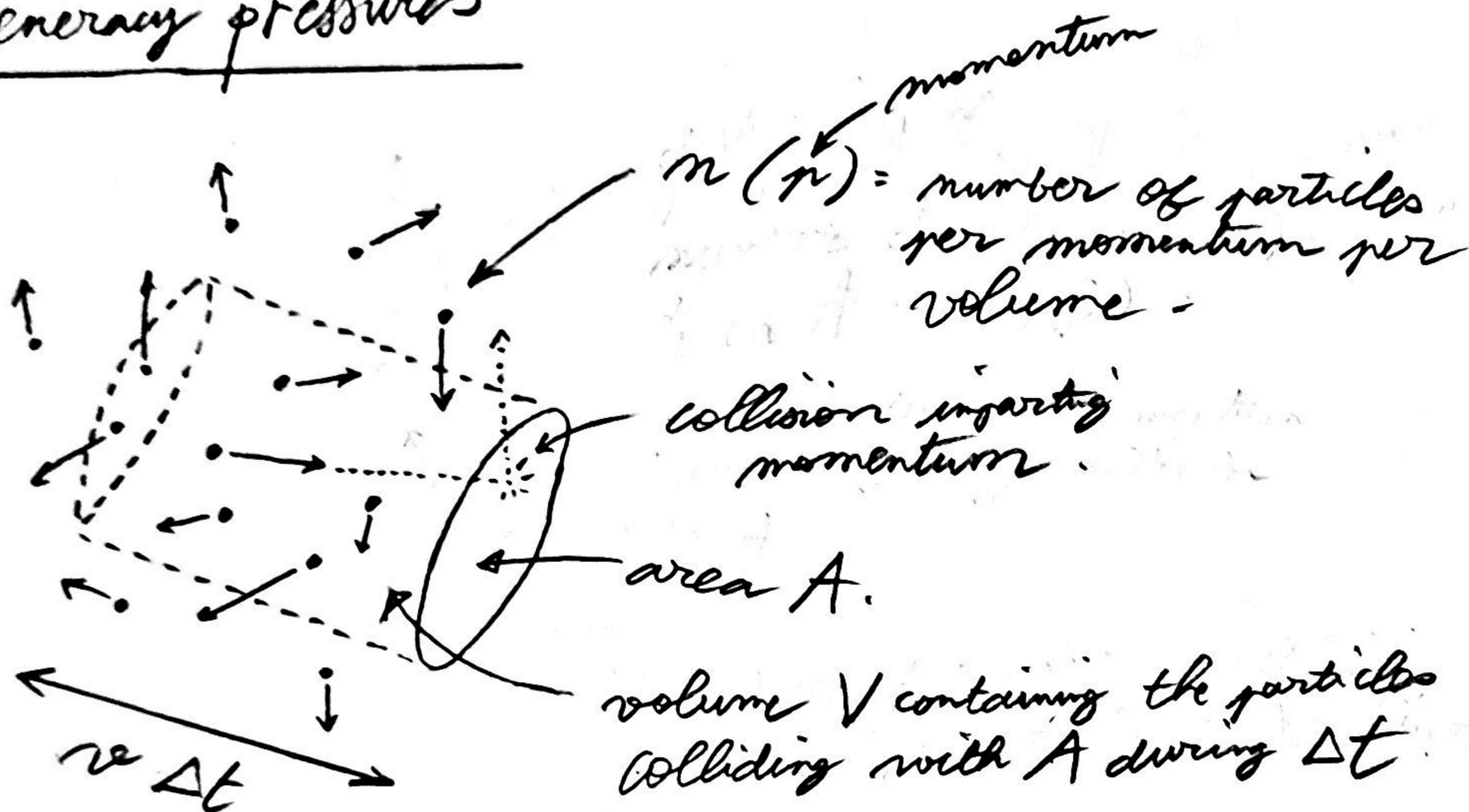
stable if $F_g' > F_p' \Rightarrow 4-3\delta < 1 \Rightarrow \boxed{\delta > \frac{4}{3}}$

to make the
star contract
back to equilibrium

condition for
dynamical
stability

($\delta = \frac{5}{3} > \frac{4}{3}$ for an ideal gas)

Degeneracy pressures.



$$P = \frac{F}{A} = \frac{\Delta p}{\Delta t} \cdot \frac{1}{A} \approx \int_0^{+\infty} p n(p) v dp.$$

momentum imparted by all collisions during Δt

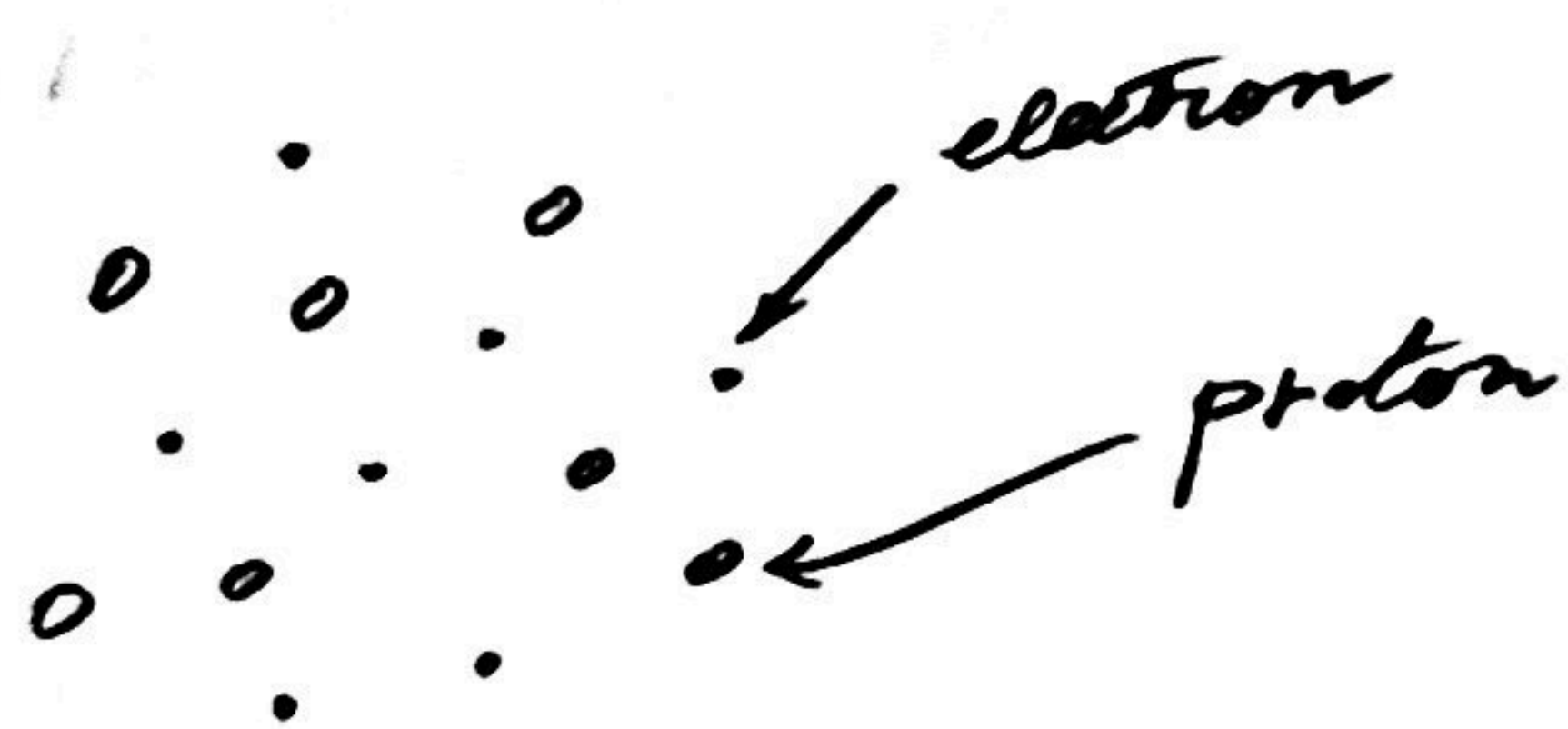
$$\Delta p \approx \int_0^{+\infty} p n(p) \underbrace{V}_{\substack{\text{momentum} \\ \text{imparted} \\ \text{by one} \\ \text{collision}}} dp$$

$V = A v \Delta t$

$$\boxed{P = \left(\frac{1}{3}\right) \int_0^{+\infty} v n(p) p dp}$$

factor
accounting
for the angles
of different
collisions
(assume isotropic
velocities)

general formula
for pressure.



Heisenberg uncertainty principle:

$$\Delta V \Delta p^3 \gtrsim h^3$$

Sets a limit to how low the momentum of particles can get, resulting in degeneracy pressure.

classical momentum of electron

$$p \approx m_e v \approx (3 m_e k T)^{1/2}$$

$$v \approx \sqrt{\frac{3 k T}{m_e}}$$

(from Maxwell-Boltzmann distribution)

$$\leq \frac{h}{(\Delta V)^{1/3}} \approx h \left(\frac{\rho}{m_p} \right)^{1/3}$$

for degeneracy pressure to be important relative to thermal pressure.

$$\Delta V \approx \frac{m_p}{\rho}$$

protons dominate the mass.

$$\text{So } \rho \gtrsim \rho_{\text{cr}}^{\text{deg}} \approx \frac{(3 m_e k T)^{3/2}}{h^3} m_p$$

For the Sun's core temperature
 $T \approx 10^7 \text{ K}$

$$\rho_{\text{cr}}^{\text{deg}} \approx (10^{-27} \times 10^{-16} \times 10^7)^{3/2} \times \frac{10^{-24}}{(10^{-27})^3}$$

$$\approx 10^{-36 \times \frac{3}{2} - 24 + 81} = 10^3 \text{ g/cm}^3$$

(Actual value = $\rho_{\text{cr}}^{\text{deg}} \approx 10^4 \text{ g/cm}^3 \gg 100 \text{ g/cm}^3$)

i.e. keeping all numerical factors.

Sun's core density, so it is not degenerate.

Now, computing the degeneracy pressure.

$$n_e(p) dp = \frac{2}{\Delta V} = \frac{2}{h^3} \Delta^3 p = \frac{8\pi}{h^3} p^2 dp$$

momentum space in 3D: $\Delta^3 p = 4\pi p^2 dp$

Pauli exclusion principle: only two electrons can have the same momentum.

assuming $T=0$, so $\Delta V \Delta^3 p = h^3$ (i.e. the momentum is specified by the Heisenberg uncertainty principle only).

$n_e = \int_0^{p_{\text{max}}} n_e(p) dp = \int_0^{p_{\text{max}}} \frac{8\pi}{h^3} p^2 dp = \frac{8\pi}{h^3} \frac{p_{\text{max}}^3}{3}$

$\uparrow$ electron number density

$\nwarrow$ maximum momentum

$\therefore p_{\text{max}} = \left(\frac{3h^3 n_e}{8\pi} \right)^{1/3}$

$v_e = \frac{p}{m_e}$ (non-relativistic particles)

$p_{\text{deg}}^{\text{nr}} = \frac{1}{3} \int_0^{p_{\text{max}}} n_e(p) p v_e dp$

$= \frac{8\pi}{3h^3} \frac{1}{m_e} \int_0^{p_{\text{max}}} p^4 dp$

$\frac{p_{\text{max}}^5}{5} = \frac{n_e^{5/3}}{5} \left(\frac{3h^3}{8\pi} \right)^{5/3}$

$= \frac{n_e^{5/3}}{m_e} \frac{1}{5} \left(\frac{3h^3}{8\pi} \right)^{2/3}$

$= \frac{1}{5} \frac{1}{8^{2/3}} \left(\frac{3}{\pi} \right)^{2/3} \frac{h^2}{m_e} n_e^{5/3}$

$\frac{1}{20}$

$p_{\text{deg}}^{\text{nr}} = \frac{1}{20} \left(\frac{3}{\pi} \right)^{2/3} \frac{h^2}{m_e} \left(\frac{\rho}{\mu_e m_p} \right)^{5/3}$

Non-relativistic electron degeneracy pressure

$n_e = \frac{N_e}{V} = \frac{1}{\mu_e} \frac{N_n}{V} = \frac{\rho}{\mu_e m_p}$

$N_n \approx M/m_p$

$\nwarrow$ number of electrons
 $\swarrow$ number of nucleons (protons + neutrons)

$\uparrow$ number of free electrons per nucleon
 $\uparrow$ one proton and one electron

Hydrogen = $\mu_e = 1$
 No hydrogen = $\mu_e \approx 2$

- equal number of protons and free electrons (charge neutral and fully-ionized gas)
- typically similar number of protons and neutrons.

$\therefore p_{\text{deg}}^{\text{nr}} = K \rho$

$\nwarrow$ $\gamma = 5/3$ (dynamically stable)

$K = \frac{1}{20} \left(\frac{3}{\pi} \right)^{2/3} \frac{h^2}{m_e} \frac{1}{(\mu_e m_p)^{5/3}} \equiv K_{\text{deg}}^{\text{nr}}$

$$P_{deg}^r = \frac{1}{3} \int_0^{p_{max}} m_e(r) p v_e dp \quad v_e \approx c \text{ (relativistic particles)}$$

$$= \frac{c}{3} \frac{8\pi}{h^3} \int_0^{p_{max}} p^3 dp$$

$$\frac{p_{max}}{4} = \frac{m_e^{4/3}}{4} \left(\frac{3h^3}{8\pi} \right)^{1/3}$$

$$= \frac{c}{4} \left(\frac{3h^3}{8\pi} \right)^{1/3} m_e^{4/3}$$

$$P_{deg}^r = \frac{hc}{8} \left(\frac{3}{\pi} \right)^{1/3} \left(\frac{\rho}{\mu_e m_p} \right)^{4/3}$$

Relativistic electron degeneracy pressure.

$$P_{deg}^r = K \rho^\gamma \leftarrow \gamma = 4/3 \text{ (dynamically unstable)}$$

$$K = \frac{hc}{8} \left(\frac{3}{\pi} \right)^{1/3} \frac{1}{(\mu_e m_p)^{4/3}} \equiv K_{deg}^r$$

$$p \gtrsim m_e c \quad \longrightarrow \quad \rho \gtrsim \frac{p^3 m_p}{h^3} \gtrsim \rho_{cr}^{rel} \approx \left(\frac{m_e c}{h} \right)^3 m_p$$

for particles to be relativistic.

$$p \approx \frac{h}{(\Delta V)^{1/3}} \approx h \left(\frac{\rho}{m_p} \right)^{1/3}$$

momentum of a degenerate particle given by the Heisenberg uncertainty principle

$$\rho_{cr}^{rel} \approx \left(\frac{10^{-27} \times 10^{10}}{10^{-27}} \right)^3 \times 10^{-24} \approx 10^{30-24} \approx 10^6 \text{ g/cm}^3$$

(close to actual value)

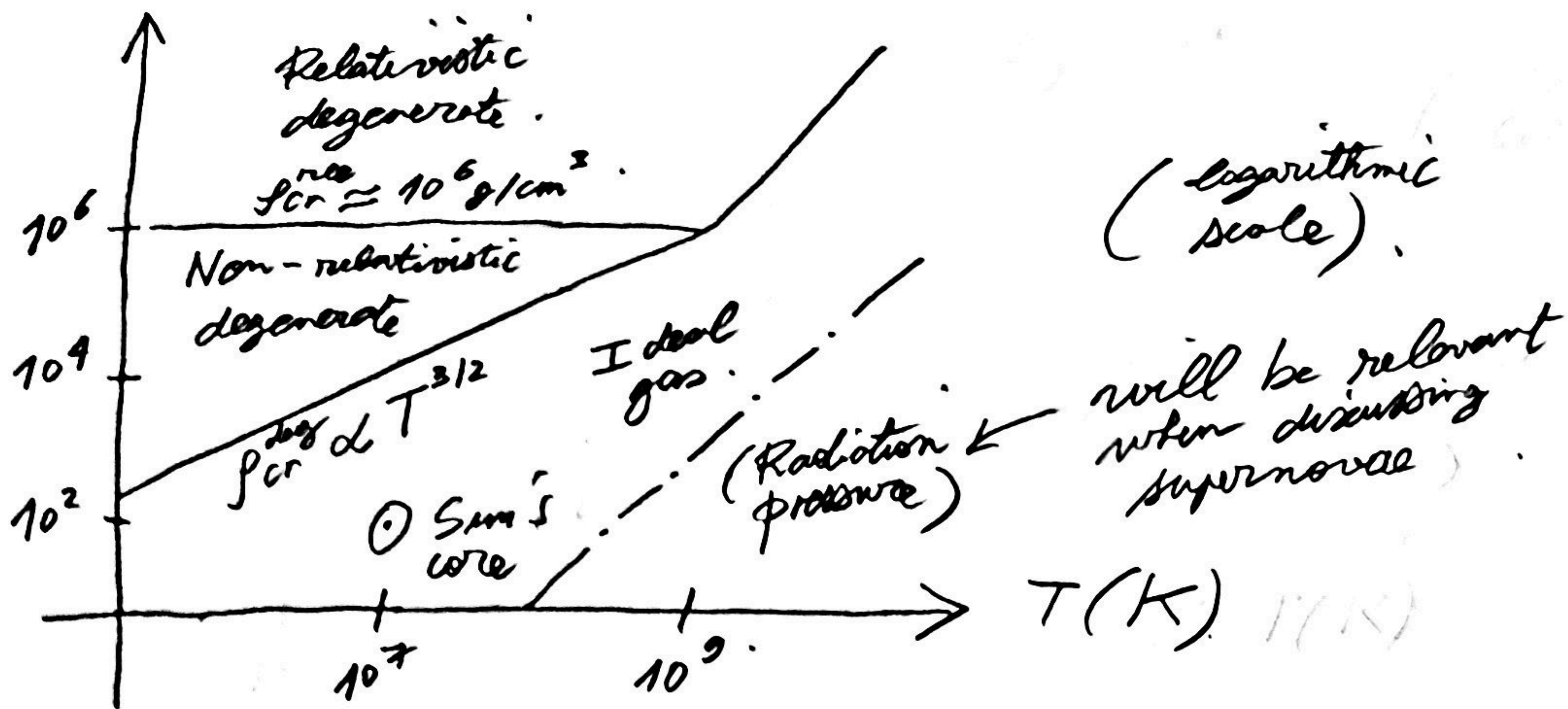
For neutrons to become relativistic:

$$\rho \gtrsim \rho_{cr}^{rel}(\text{neutrons}) = \left(\frac{m_p}{m_e} \right)^3 \rho_{cr}^{rel}(\text{electrons})$$

$$\approx (1000)^3 10^6$$

$$\approx 10^{15} \text{ g/cm}^3$$

$\rho (\text{g/cm}^3)$



Polytropic models.

Hydrostatic equilibrium: $\frac{1}{\rho} \frac{dP}{dR} = \frac{GM}{R^2}$

order of magnitude $\rightarrow$

$$\frac{1}{\rho} \frac{P}{R} \approx K \left(\frac{M}{R^3} \right)^{\delta-1} \frac{1}{R}$$

$$\begin{cases} \cdot P = K \rho^\delta \\ \cdot \rho \approx M/R^3 \end{cases}$$

$$K \left(\frac{M}{R^3} \right)^{\delta-1} \approx \frac{GM}{R} \Rightarrow KM^{\delta-2} \approx GR^{3\delta-4}$$

$$\text{So: } \boxed{M \approx \left(\frac{G}{K} \right)^{\frac{1}{\delta-2}} R^{\frac{3\delta-4}{\delta-2}}}$$

Mass-radius relation for a polytrope.

Chandrasekhar mass limit for white dwarfs.

Ideal gas = $M \propto R^{\frac{1}{-1/3}} = R^{-3} \Rightarrow \rho \approx \frac{M}{R^3} \propto M^2$
 $\delta = 5/3$

If M increases, so does ρ until $\rho > \rho_{cr} \Rightarrow \delta = 4/3$.
 (electrons become degenerate and relativistic)

Relativistic
degenerate
electrons:

$$M \approx \left(\frac{G}{K} \right)^{\frac{1}{\frac{4}{3}-2}} R^0 = \left(\frac{K}{G} \right)^{3/2}$$

$\delta = 4/3$
in mass-radius relation.

No solution for hydrostatic
equilibrium beyond this mass.

$$M \approx \left(\frac{hc}{G m_p^{4/3}} \right)^{3/2} \mu e^{-2}$$

$$K = K_{\text{deg}} \approx \frac{hc}{(\mu m_p)^{4/3}}$$

$$M_3 \approx 2.02$$

(comes from solving the hydrostatic equilibrium equation numerically)

Actual value:

$$M_{\text{Ch}} = \frac{M_3}{4\pi} \left(\frac{3}{2} \right)^{1/2} \left(\frac{hc}{G m_p^{4/3}} \right)^{3/2} \mu e^{-2}$$

Chandrasekhar mass
limit.

$$M_{\text{Ch}} \approx \frac{5.83}{\mu e^2} M_{\odot} = 1.4 M_{\odot}$$

value for
white dwarfs.
($\mu e = 2$ because
no hydrogen)

Physical interpretation:

$$\frac{F_p}{F_g} \approx \frac{P/\rho R}{GM/R^2} \approx \frac{K_{\text{deg}}^{\text{rel}}}{G} \left(\frac{(M/R^3)^{1/3}}{M/R} \right) \approx \left(\frac{M}{M_{\text{Ch}}} \right)^{-2/3}$$

$\left\{ \begin{array}{l} P = K_{\text{deg}}^{\text{rel}} \rho^{4/3} \\ \rho \approx M/R^3 \end{array} \right.$

$\frac{K_{\text{deg}}^{\text{rel}}}{G} \approx M_{\text{Ch}}^{2/3}$

If $M > M_{\text{Ch}}$,

$$\frac{F_p}{F_g} < 1$$

(not enough
pressure to
support
gravity)

Gravitational
collapse to a
neutron star
or black hole.

This happens for:

- White dwarfs gaining mass from a binary companion.
- Iron cores forming at the end of a massive star's evolution.

Estimating the radius:

$$p \gtrsim m_e c \quad \xrightarrow{\text{for electrons to be relativistic}} \quad R \lesssim R_{\text{rel}} \approx \left(\frac{M}{m_p} \right)^{1/3} \frac{h}{m_e c}$$

$p \approx \frac{h}{\Delta V} \approx h \left(\frac{\rho}{m_p} \right)^{1/3}$
 uncertainty principle
 $\rho \approx \frac{m_p}{\Delta V}$
 (because protons dominate the mass)
 $\rho \approx \frac{M}{R^3}$ total mass / total radius

$M \approx M_{\text{Ch}} \approx 1 M_{\odot}$

$$R_{\text{rel}} \approx \left(\frac{10^{33}}{10^{-24}} \right)^{1/3} \frac{10^{-27}}{10^{-27} 10^0} \approx 10^{19-10} \approx 10^9 \text{ cm} \approx 10^{-2} R_{\odot} \equiv R_{\text{WD}}$$

$$\rho_{\text{WD}} \approx \frac{M_{\text{Ch}}}{R_{\text{WD}}^3} \approx \frac{10^{33}}{(10^9)^3} = 10^6 \text{ g/cm}^3$$

$R_{\odot} \approx 10^{11} \text{ cm}$ (consistent with the radius of white dwarfs)

Neutron stars.

Supported by the degeneracy pressure of relativistic neutrons.

$$R_{\text{rel}}(\text{neutrons}) \approx \frac{m_e}{m_p} \times R_{\text{rel}}(\text{electrons}) \approx 10^{-3} \times 10^9 \text{ cm} = 10 \text{ km} \equiv R_{\text{NS}}$$

(consistent with the radius of neutron stars)

$$M_{\text{Ch}}(\text{neutrons}) \approx 5.83 M_{\odot}$$

$$\frac{1}{\mu_e} \rightarrow \left(\frac{1}{\mu_n} \right) = 1 \quad \left(\begin{array}{l} \text{gas made} \\ \text{of only} \\ \text{neutrons} \end{array} \right)$$

|||
number of
free neutrons
per nucleons.

But this mass limit is inaccurate because.

- A general-relativistic treatment is needed $\left(R_{\text{NS}} \lesssim 10 \frac{GM}{c^2} \right)$ assuming $M \approx M_{\odot}$
- A more complicated equation of state must be used, i.e. $P \neq K \rho^{\gamma}$.

Actual value =

$$M_{TOV} \approx 2.2 M_{\odot}$$

Tolman - Oppenheimer - Volkoff
mass limit for neutron stars.

$$\underline{M > M_{TOV} \Rightarrow}$$

Gravitational collapse to
a black hole.

$$\rho_{NS} \approx \frac{M_{TOV}}{R_{NS}^3} \approx \frac{10^{32}}{(10^6)^3} \approx 10^{15} \text{ g/cm}^3.$$

$$\text{Weak interactions: } \begin{cases} e^- + p \rightarrow n + \nu & (\text{inverse } \beta\text{-decay}) \\ n \rightarrow p + e^- + \bar{\nu} & (\beta\text{-decay}). \end{cases}$$

Neutrons get formed when β -decay is suppressed, which happens when the electrons produced have a momentum corresponding to an already occupied quantum state.

$$p \leq p_{max}$$

momentum
of produced
electron

momentum
below which
all quantum
states are occupied.

$$\begin{cases} \cdot n_e = \frac{f}{\mu_{emp}} \\ \cdot \mu_e \approx 1 \end{cases}$$

$$\begin{cases} \cdot p_{max} = \left(\frac{3 h^3 m_e}{8 \pi} \right)^{1/3} \approx \left(\frac{3 h^3 f}{8 \pi m_p} \right)^{1/3} \\ \cdot p = \left(\frac{E^2}{c^2} - m_e^2 c^2 \right)^{1/2} \quad (\text{for relativistic electrons}) \end{cases}$$

$$E^2 - m_e^2 c^4 \leq \left(\frac{3 h^3 f}{8 \pi m_p} \right)^{2/3} c^2.$$

energy of
electrons:

$$E \geq (m_n - m_p) c^2 \equiv \Delta E_{np}$$

neutron
mass

for inverse
 β -decay to
proceed.

$$\Delta E_{np}^2 - m_e^2 c^4 \lesssim \left(\frac{3 h^3 \rho}{8 \pi m_p} \right)^{2/3} c^2.$$

$$\rho \gtrsim \frac{8 \pi m_p}{3 h^3 c^3} \left(\Delta E_{np}^2 - m_e^2 c^4 \right)^{3/2}.$$

$$\approx \underbrace{\frac{8 \pi m_p m_e^3 c^3}{3 h^3}}_{\substack{\approx \\ 10^6}} \underbrace{\left[\left(\frac{\Delta E_{np}}{m_e c^2} \right)^2 - 1 \right]^{3/2}}_{\substack{\approx \\ 10}} \approx 10^7 \text{ g/cm}^3.$$

Additional complications:

- Neutron drip = neutron-rich nuclei break up, liberating free neutrons ($\rho_{nd} \approx 4 \times 10^{11} \text{ g/cm}^3$).
- The strong force starts dominating around nuclear densities ($\rho_{mc} \approx 3 \times 10^{14} \text{ g/cm}^3$).

These processes complicate the determination of the equation of state, which becomes uncertain close to the core.