

General - relativistic effects .

The exterior of non - rotating compact object can be described in general relativity by the Schwarzschild metric :

$$ds^2 \equiv c^2 d\tau^2 = \left(1 - \frac{2R_g}{r}\right) c^2 dt^2 - \left(1 - \frac{2R_g}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

$\tau \equiv$ proper time .
(measured by a clock
that follows the path
in space - time described
by the coordinates)

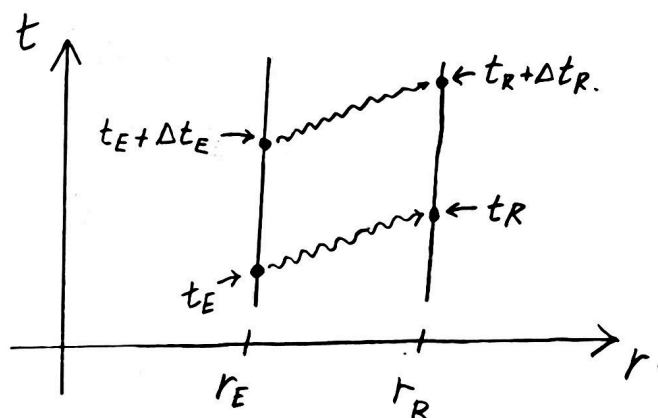
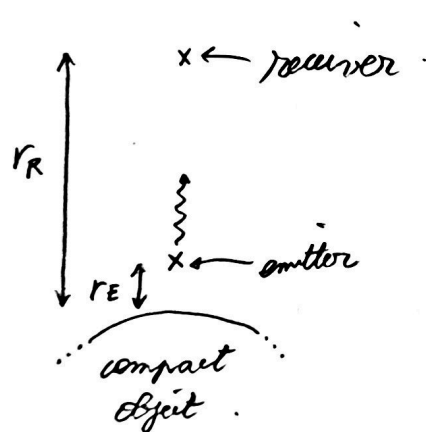
$ct \equiv$ coordinate
time .

spherical
coordinates
(r, θ, ϕ) .

$$R_g \equiv \frac{GM}{c^2} \leftarrow \text{mass of the central object ,}$$

(gravitational radius)

Gravitational redshift .



(fixed emitter
and receiver)

$$t_R - t_E = (t_R + \Delta t_R) - (t_E + \Delta t_E) \Rightarrow \Delta t_R = \Delta t_E$$

photons emitted at t_E
and $t_E + \Delta t_E$ cover the
same distance moving at
the same speed .

$$dr = d\theta = d\phi \text{ (fixed) .}$$

For both emitter
and receiver :

$$d\tau^2 = \left(1 - \frac{2R_g}{r}\right) dt^2$$

Integrating:
$$\begin{cases} \Delta\tau_E = \left(1 - \frac{2R_g}{r_E}\right)^{1/2} \Delta t_E \\ \Delta\tau_R = \left(1 - \frac{2R_g}{r_R}\right)^{1/2} \Delta t_R \end{cases}$$

$\Downarrow \leftarrow \Delta t_E = \Delta t_R$

$\left(\Delta\tau_E / \Delta\tau_R = \text{time intervals measured in a frame at rest with respect to the emitter/receiver} \right)$

$$\frac{\Delta\tau_R}{\Delta\tau_E} = \left(\frac{1 - 2R_g/r_R}{1 - 2R_g/r_E} \right)^{1/2}$$

So:
$$\frac{\nu_R}{\nu_E} = \left(\frac{1 - 2R_g/r_E}{1 - 2R_g/r_R} \right)^{1/2}$$

$\nu = 1/\Delta\tau$
(frequency of signal)

Gravitational redshift for a fixed emitter and receiver.

Application = shift in frequency for an atomic line emitted from the surface of a compact object and observed at infinity.

$$\frac{\Delta\nu}{\nu_0} = 1 - \frac{\nu_R}{\nu_E} = 1 - \left(1 - \frac{2R_g}{R}\right)^{1/2} \approx \frac{GM}{c^2 R}$$

$\left\{ \begin{array}{l} \cdot \nu_E = \nu_0 \text{ (frequency of the atomic line)} \\ \cdot \nu_R = \nu_0 - \Delta\nu \text{ (redshifted frequency)} \end{array} \right.$

$\left\{ \begin{array}{l} \cdot r_R = +\infty \\ \cdot r_E = R \end{array} \right.$
 $\uparrow$
 radius of the compact object.

$\frac{R_g}{R} \ll 1$
 $(R_g = GM/c^2)$

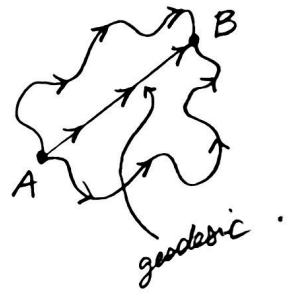
Can be used to measure R , knowing $M, \Delta\nu, \nu_0$.

Trajectory of particles.

Particles follow geodesics obtained by minimizing the action:

$$S = \int_A^B ds = \int_{\tau_A}^{\tau_B} \tilde{\mathcal{L}}(x^\mu, \dot{x}^\mu) d\tau \quad \leftarrow \text{proper time}$$

two events in space-time



$$\begin{cases} x^\mu = (ct, r, \theta, \phi) \\ \dot{x}^\mu = dx^\mu/d\tau \end{cases}$$

(coordinates describing a path in space-time going from event A to B.)

Lagrangian

$$\mathcal{L} \equiv \tilde{\mathcal{L}}^2 = \left(\frac{ds}{d\tau}\right)^2 = \left(1 - \frac{2R_g}{r}\right) c^2 \dot{t}^2 - \left(1 - \frac{2R_g}{r}\right)^{-1} \dot{r}^2 - r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)$$

$$\delta S = 0 \implies \frac{d}{d\tau} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \right) - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0.$$

(Euler-Lagrange equations)

differentiating with respect to coordinates.

Assuming: $\begin{cases} \theta = \pi/2 \\ \dot{\theta} = 0 \end{cases} \implies \mathcal{L} = \left(1 - \frac{2R_g}{r}\right) c^2 \dot{t}^2 - \left(1 - \frac{2R_g}{r}\right)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2$

(without loss of generality)

$$x^0 = t: \frac{d}{d\tau} \left(\frac{\partial \mathcal{L}}{\partial \dot{t}} \right) = 0 \implies \left(1 - \frac{2R_g}{r}\right) \dot{t} \equiv k = \frac{E}{mc^2}$$

constant of motion

energy of particle

rest mass energy

$$x^3 = \phi: \frac{d}{d\tau} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = 0 \implies r^2 \dot{\phi} = h \leftarrow \text{constant angular momentum}$$

$$\left(\frac{ds}{d\tau}\right)^2 = c^2 \implies \left(1 - \frac{2R_g}{r}\right) c^2 \dot{t}^2 - \left(1 - \frac{2R_g}{r}\right)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2 = c^2$$

$\Downarrow \leftarrow$ multiplying by $1 - 2R_g/r$

$$k^2 c^2 - \dot{r}^2 - \left(1 - \frac{2R_g}{r}\right) \frac{h^2}{r^2} = c^2 \left(1 - \frac{2R_g}{r}\right)$$

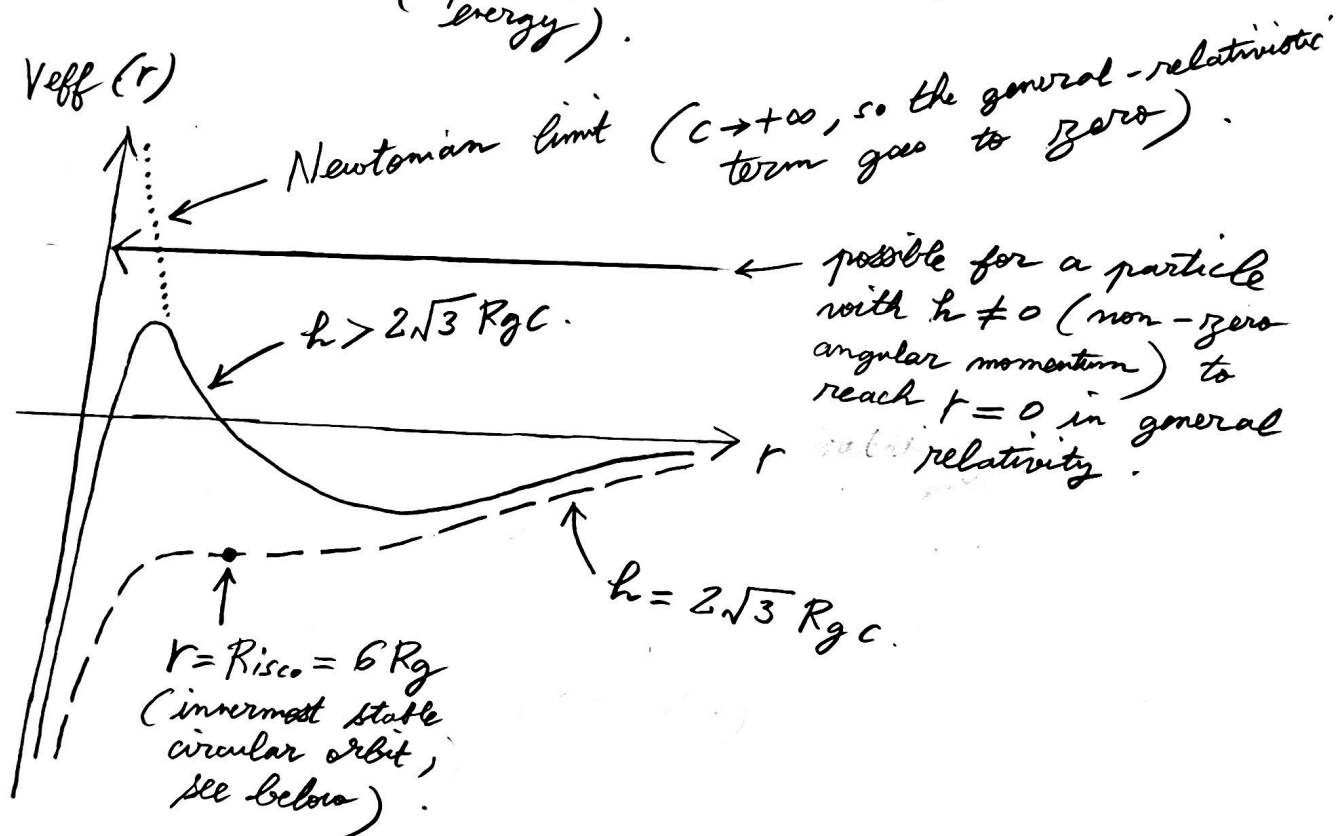
$$\dot{r}^2 + \frac{h^2}{r^2} - \frac{2R_g c^2}{r} - \frac{2R_g h^2}{r^3} = (h^2 - 1)c^2.$$

So:

$$\boxed{\frac{1}{2} \dot{r}^2 + V_{\text{eff}}(r) = (h^2 - 1)c^2/2.}$$

$$V_{\text{eff}}(r) = \frac{h^2}{2r^2} - \frac{R_g c^2}{r} - \frac{R_g h^2}{r^3}.$$

centrifugal barrier. $\frac{h^2}{2r^2}$
 $\frac{R_g c^2}{r} \parallel \frac{-GM}{r}$ (potential energy).
 $\frac{R_g h^2}{r^3}$ general-relativistic term. (Effective potential for massive particles)



Stability of orbits.

$$\frac{dV_{\text{eff}}}{dr} = -\frac{h^2}{r^3} + \frac{R_g c^2}{r^2} + 3\frac{R_g h^2}{r^4}$$

$$\frac{dV_{\text{eff}}}{dr} = 0 \Rightarrow R_g c^2 r^2 - h^2 r + 3R_g h^2 = 0$$

to find minimum and maximum.

$$\Delta = h^4 - 4 \times 3 R_g^2 c^2 h^2 = h^2 (h^2 - 12 R_g^2 c^2)$$

minimum (radius of circular orbits).

$$r_{\pm} = \frac{h^2 \pm \sqrt{\Delta}}{2 R_g c^2} = \frac{h}{2 R_g c^2} \cdot \left(h \pm \sqrt{h^2 - 12 R_g^2 c^2} \right)$$

maximum (unstable)

$$r_+ = r_- \Rightarrow h = \sqrt{12} R_g c = 2\sqrt{3} R_g c \quad \left(\text{marginal stability at only one radius} \right)$$

$$r = \frac{h^2}{2 R_g c^2} \Rightarrow r \equiv r_{isco} = 6 R_g$$

Radius of the innermost stable circular orbit.

Finding the corresponding energy:

$$\dot{r} = 0 \quad \left(\text{for a circular orbit} \right) \Rightarrow V_{\text{eff}}(r) = (k^2 - 1) c^2 / 2$$

$$\frac{h^2}{2r^2} - \frac{R_g c^2}{r} - \frac{R_g h^2}{r^3} = \left(\frac{1}{6} - \frac{1}{6} - \frac{1}{18} \right) c^2 = -\frac{1}{18} c^2$$

$$\begin{cases} \bullet h = 2\sqrt{3} R_g c \\ \bullet r = 6 R_g \end{cases} \quad \left(\frac{h^2}{r^2} = \frac{12 R_g^2 c^2}{36 R_g^2} = \frac{1}{3} c^2 \right)$$

$$\text{so: } k = \sqrt{1 - \frac{1}{9}} = \frac{2\sqrt{2}}{3}$$

energy of the particle at $r = r_{isco}$ $\rightarrow E/mc^2$

Energy lost by a particle moving from $r = +\infty$ to $r = R_{isco}$ on a series of circular orbits with decreasing angular momentum =

$$\Delta E = \underbrace{E(r = +\infty)}_{\substack{\parallel \\ mc^2 \\ \text{(only rest mass energy because no effect of gravity)}}} - \underbrace{E(r = R_{isco})}_{\parallel} = \underbrace{\left(1 - \frac{2\sqrt{2}}{3}\right)}_{\substack{\parallel \\ 0.057 = 5.7\%}} mc^2$$

$$\frac{2\sqrt{2}}{3} mc^2$$

Possible to extract $\sim 5.7\%$ of the particle's rest mass energy through this process, which is about ten times more than nuclear burning of Hydrogen ($\sim 0.7\%$). In practice, this can be achieved for gas spiralling inward in an accretion disc; the energy is then converted to radiation.

For a rotating compact object (described by the Kerr metric), $R_{isco} = R_g$ and $\Delta E \approx 42\% mc^2$ if the particle co-rotates (more efficient than the non-rotating case).

Hawking effect

Process to extract energy from a black hole.

Due to quantum fluctuations, a pair of particles can be created out of vacuum. This violates energy conservation, but is allowed by the uncertainty principle if the particles annihilate in a time $\Delta t \lesssim \frac{\hbar}{\Delta E}$, where ΔE is the amount of violation.

Let's consider two particles created near a black hole with energies $E > 0$ and $-E < 0$. While negative energy is not allowed outside the event horizon ($r > 2R_g$), it is allowed inside ($r < 2R_g$) because the energy measured by any observer in this region corresponds to a component of the particle's momentum. So, if the negative energy

particle crosses the event horizon within a proper time $\Delta\tau = \frac{\hbar}{\Delta E}$, no physical law has been broken. By adding negative energy to the black hole, we have effectively extracted energy from it. This is the Hawking effect.

The energy of the outgoing particle turns out to be $E = \frac{\hbar c^3}{8\pi G M}$.

Associated temperature = $T = \frac{\hbar c^3}{8\pi k_B G M}$ Hawking temperature

$E = k_B T$ (energy per particle)

The black hole loses energy (and therefore mass) as a black body:

$$\frac{dM}{dt} = - \frac{\sigma T^4 A}{c^2}$$

← area of the event horizon
 $A = 16\pi R_g^2$

← $\sigma = \frac{\pi^2 k_B^4}{60 \hbar^3 c^2}$ (Stefan-Boltzmann constant)

$$\frac{1}{c^2} \frac{\pi^2 k_B^4}{60 \hbar^3 c^2} \left(\frac{\hbar c^3}{8\pi k_B G M} \right)^4 \times 16\pi \left(\frac{G M}{c^2} \right)^2 = \alpha \hbar / M^2$$

$$\alpha = \frac{c^{12-4-2-2}}{\pi \times \left(\frac{60 \times 8^4}{16} \right) G^2} = \frac{c^4}{15360 \pi G^2}$$

11 55

15360 4×10^{51} (cgs)

So: $\boxed{\frac{dM}{dt} = - \frac{\alpha \hbar}{M^2}}$ Mass loss due to black hole evaporation

$$\tau_{\text{evap}} \approx \frac{M}{\left| \frac{dM}{dt} \right|} = \frac{M^3}{\alpha \hbar} \approx \frac{(2 \times 10^{33})^3}{4 \times 10^{51} \times 10^{-27}} \left(\frac{M}{M_0} \right)^3$$

11 75

$2 \times 10^{75} \text{ s} \approx 7 \times 10^{67} \text{ yr}$

For primordial black holes formed at the Big Bang to still be present today:

$$\tau_{\text{evap}} > \tau_{\text{Hubble}} \approx 13.5 \text{ Gyr} \quad (\text{age of the Universe}).$$



$$M > \left(\frac{13.5 \text{ Gyr}}{7 \times 10^{67} \text{ yr}} \right)^{1/3} M_{\odot} \approx 10^{14} \text{ g} \quad (\text{mass of an asteroid}).$$

$\approx 2 \times 10^{33} \text{ g}$