

Gravitational waves

Propagating perturbations of space-time that can be produced by the motion of compact objects.

First directly observed in 2015 by the "Laser Interferometer Gravitational-wave Observatory" (LIGO).

Existence of gravitational waves

Ricci
tensor
↓
 $R_{\mu\nu}$

metric
tensor
↓
 $g_{\mu\nu}$

Ricci
scalar
↓
 R

stress-
energy
tensor
↓
 $T_{\mu\nu}$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = - \frac{8\pi G}{c^4} T_{\mu\nu}$$

Einstein's gravitational field equations

Minkowski metric
(flat space-time)
↓
 $\eta_{\mu\nu}$

small perturbation
($|h_{\mu\nu}| \ll 1$)
↓
 $h_{\mu\nu}$

Writing $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and linearizing leads to:

$$\underbrace{\square}_{\equiv \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2} h^{\mu\nu} = - \frac{16\pi G}{c^4} T^{\mu\nu} \quad \text{for} \quad \underbrace{\partial_\mu h^{\mu\nu} = 0}_{\text{Lorenz gauge condition}}$$

Lorenz gauge condition (can be imposed because it does not modify Einstein's field equations)

Similar to the equations for the electromagnetic potential:

4-potential
 $A^\mu = (\phi, \vec{A})$
↑
electrostatic potential
 ϕ
3-vector potential
 $\vec{A}$

4-current
 $j^\mu = (c\rho, \vec{j})$
↑
charge density
 ρ
3-current density
 $\vec{j} = \rho \vec{v}$

$$\square A^\mu = \frac{4\pi}{c} j^\mu \quad \text{for} \quad \partial_\mu A^\mu = 0$$

Lorenz gauge condition (can be imposed because it does not modify the electromagnetic field equations)

Gravitational waves are analogous to electromagnetic waves with electric charge replaced by mass.

Propagation and polarization states

In vacuum: $\vec{\square}^2 h^{\mu\nu} = 0 \Rightarrow h^{\mu\nu} = A^{\mu\nu} \exp(i k_\rho x^\rho)$.

$\uparrow$ $T^{\mu\nu} = 0$ $\nwarrow$ general solution
 $\uparrow$ $k^\sigma = \eta^{\sigma\rho} k_\rho = (\frac{\omega}{c}, \vec{k})$ $\swarrow$ angular frequency
 $\uparrow$ 4-wavevector $\uparrow$ 3-wavevector.

$$\vec{\square}^2 h^{\mu\nu} = 0 \Rightarrow \eta^{\sigma\rho} k_\sigma k_\rho = 0 \Rightarrow \left(\frac{\omega}{c}\right)^2 - \vec{k}^2 = 0.$$

$$\vec{\square}^2 = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 = \eta^{\sigma\rho} \frac{\partial}{\partial x^\sigma} \frac{\partial}{\partial x^\rho}$$

So: $\omega = \pm c|\vec{k}|$, implying that gravitational waves propagate at the speed of light.

Let's assume $k^\sigma = (k, 0, 0, k)$, that is a wave propagating in the z-direction. There are three conditions on the amplitude $A^{\mu\nu}$:

(1) Symmetry: $A^{\mu\nu} = A^{\nu\mu}$.

(2) Lorenz gauge condition: $\partial_\mu h^{\mu\nu} = 0 \Rightarrow A^{0\mu} = A^{3\mu}$.

$$\begin{aligned} & \underbrace{i k_\mu A^{\mu\nu} \exp(i k_\rho x^\rho)}_{=} \\ & \underbrace{\eta_{\mu\sigma} k^\sigma A^{\mu\nu}}_{=} = k A^{0\nu} - k A^{3\nu} \\ & \underbrace{(k, 0, 0, -k)}_{=} \end{aligned}$$

(3) Additional gauge transformation: $h_{\mu\nu} \rightarrow h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$.
(not affecting the field equations)

$\nwarrow$ arbitrary functions.

Can be chosen such that,

$$\begin{cases} \cdot A^{30} = A^{31} = A^{32} = 0 \text{ ("transverse")} \\ \cdot A^{00} + A^{11} + A^{22} + A^{33} = 0 \text{ ("traceless")}. \end{cases}$$

In this «transverse-traceless» gauge, the amplitude can be written:

$$A^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \left(\begin{array}{l} \text{only 2 independent components,} \\ \text{or «polarization» states in} \\ \text{analogy with electromagnetism} \end{array} \right)$$

Effect on free particles.

Consider a free particle at rest with 4-velocity $\frac{dx^\mu}{d\tau} = (c, \vec{0})$. Its 4-acceleration during the passage of a gravitational wave is:

$$\frac{d^2 x^\mu}{d\tau^2} = - \underbrace{\Gamma^\mu_{\nu\sigma}}_{\text{Christoffel symbols}} \frac{dx^\nu}{d\tau} \frac{dx^\sigma}{d\tau} = - \Gamma^\mu_{00} c^2$$

Geodesics equation
(see lecture on general relativity)

«Christoffel symbols»

$$\Gamma^\mu_{\nu\sigma} \equiv \frac{1}{2} g^{\mu\kappa} (\partial_\nu g_{\kappa\sigma} + \partial_\sigma g_{\nu\kappa} - \partial_\kappa g_{\nu\sigma})$$

$$\Gamma^\mu_{00} = \frac{1}{2} g^{\mu\kappa} (\partial_0 g_{\kappa 0} + \partial_0 g_{0\kappa} - \partial_\kappa g_{00}) = 0$$

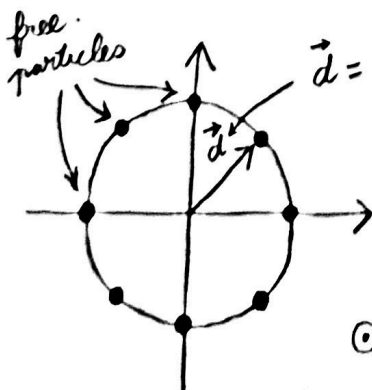
$$\parallel \leftarrow g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$\partial_0 h_{\kappa 0} + \partial_0 h_{0\kappa} - \partial_\kappa h_{00} = 0$$

$$\uparrow$$

$$A^{\mu\nu} = 0 \Rightarrow h^{\mu\nu} = 0$$

So, $\frac{d^2 x^\mu}{d\tau^2} = 0$, meaning that the particle remains at rest in this coordinate system.



$\vec{d} = (d^1, d^2, d^3)$: coordinates separation between any two particles in flat space.

⊙ ← direction of gravitational wave propagation

$$l^2 = (d^1)^2 + (d^2)^2 + (d^3)^2 = -\eta_{ij} d^i d^j = \delta_{ij} d^i d^j$$

$\uparrow$ $l \equiv$ proper distance between particles.

$$\delta_{ij} =$$

$\vec{\xi} = (\xi^1, \xi^2, \xi^3)$: coordinates separation in curved space.

$$l^2 = -\underbrace{g_{ij}}_{\substack{\eta_{ij} + h_{ij} \\ -\delta_{ij}}} \xi^i \xi^j = (\underbrace{\delta_{ij}}_{\substack{\text{variable} \\ \text{in time}}} - \underbrace{h_{ij}}_{\text{constant}}) \xi^i \xi^j$$

$\Downarrow$
 $l^2 = \delta_{ij} d^i d^j$ is variable in time and d as well since δ_{ij} is constant.

We can write: $d^i = \xi^i + \epsilon^i$
 $\epsilon^i \ll \xi^i$ (perturbation to the coordinates separation).

$$l^2 = \delta_{ij} d^i d^j = \delta_{ij} (\xi^i + \epsilon^i) (\xi^j + \epsilon^j)$$

at first order in $|\epsilon^i| \ll 1$ $\approx \delta_{ij} \xi^i \xi^j + 2 \delta_{ij} \epsilon^i \xi^j$

$$2 \delta_{ij} \epsilon^i \xi^j = -h_{ij} \xi^i \xi^j \Rightarrow 2 \delta_{ij} \epsilon^i = -h_{ij} \xi^j$$

$\uparrow$ because l^2 is independent of the choice of coordinates

$$\begin{aligned} & -\delta^{jk} \\ & \Downarrow \leftarrow \text{multiplying by } \eta^{jk} \\ & 2 \underbrace{\eta^{jk} \delta_{ij}}_{-\epsilon^k} \epsilon^i = - \underbrace{\eta^{jk} h_{ij}}_{h_i^k} \xi^j \end{aligned}$$

$$\epsilon^i = \frac{1}{2} h_k^i \xi^k \quad (\text{swapping indices } i \text{ and } k)$$

$$\text{So: } d^i = \xi^i + \frac{1}{2} h_k^i \xi^k$$

* Polarization state with $a \neq 0$ and $b = 0$:

$$d^i = \xi^i + \frac{1}{2} \underbrace{h^i_k}_{\parallel} \xi^k = \xi^i - \frac{1}{2} \delta_{kj} \underbrace{A^{ij}}_{\parallel} \cos[k(ct-z)] \xi^k.$$

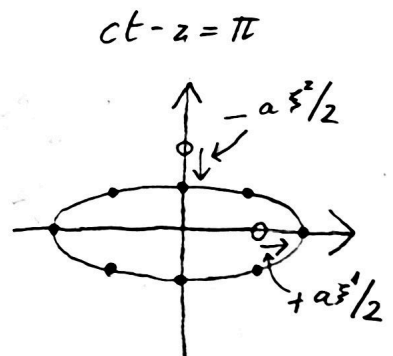
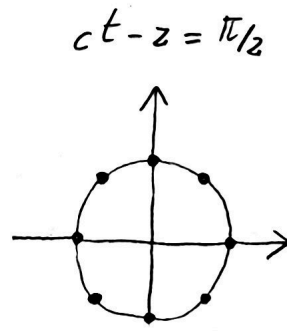
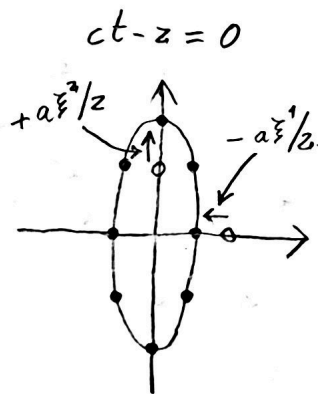
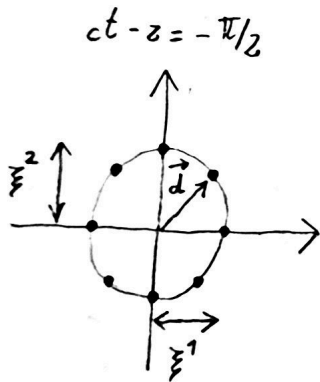
$$-\delta_{kj} = \{ \eta_{kj} \underbrace{h^{ij}}_{\parallel}$$

taking the
real part
of the general
solution

$$A^{ij} \cos[k(ct-z)].$$

$$\begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$So: \begin{cases} d^1 = \xi^1 - \frac{1}{2} a \xi^1 \cos[k(ct-z)] \\ d^2 = \xi^2 + \frac{1}{2} a \xi^2 \cos[k(ct-z)] \\ d^3 = \xi^3 \end{cases}$$



($\xi^1 = \xi^2$ for a
circular ring)

(back to
circular)

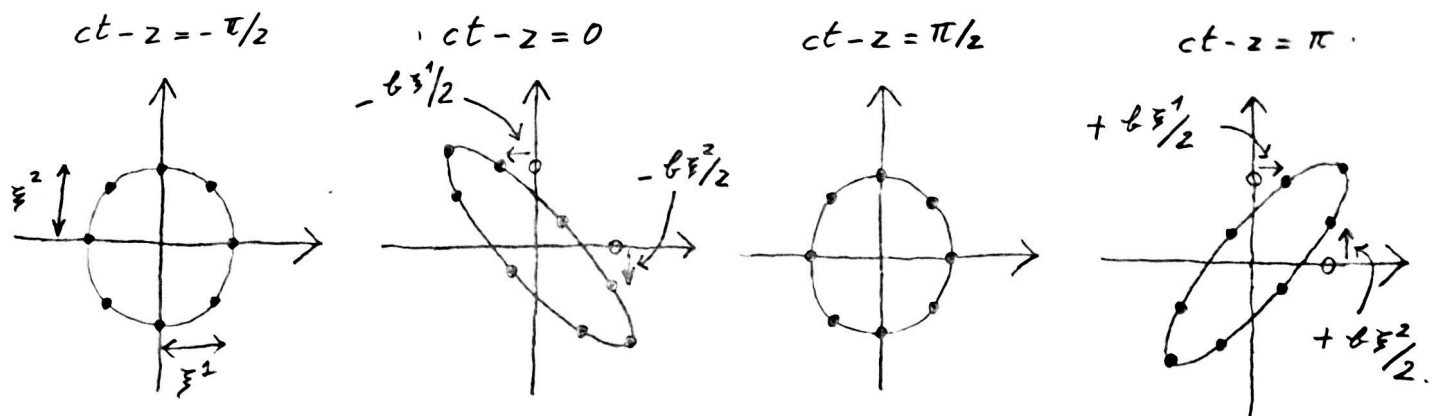
* Polarization state with $a = 0$ and $b \neq 0$:

$$d^i = \xi^i - \frac{1}{2} \delta_{kj} \underbrace{A^{ij}}_{\parallel} \cos[k(ct-z)] \xi^k$$

same as
above

$$\begin{pmatrix} 0 & b & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

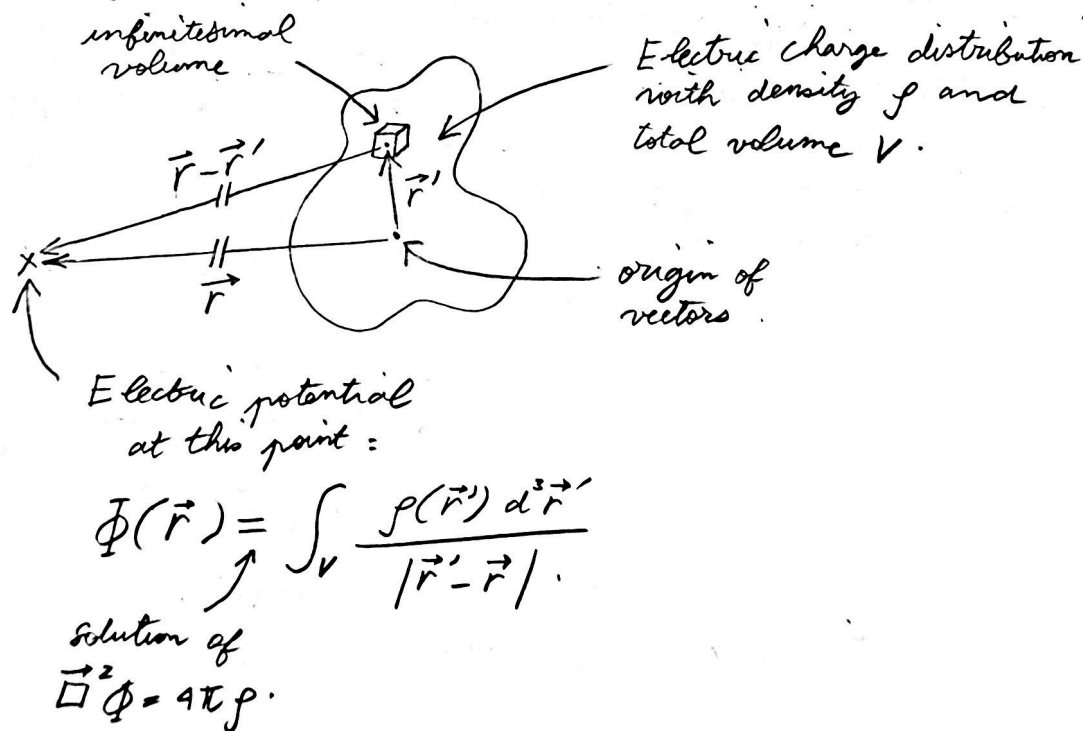
$$So: \begin{cases} d^1 = \xi^1 - \frac{1}{2} b \xi^2 \cos[k(ct-z)] \\ d^2 = \xi^2 - \frac{1}{2} b \xi^1 \cos[k(ct-z)] \\ d^3 = \xi^3 \end{cases}$$



A gravitational wave has different effects on free particles depending on its polarization state.

Production of gravitational waves

We will use the analogy with electromagnetic radiation to describe how gravitational waves are generated.



This situation is equivalent to a mass distribution through the substitution:

$$q \text{ (electric charge)} \rightarrow Gm \text{ (gravitational constant times mass)}$$

Assuming that $|\vec{r}'|/|\vec{r}| \ll 1$ (evaluating the potential at a distance much larger than the size of the charge distribution), the potential can be developed into a "multipole expansion".

$$\frac{1}{|\vec{r}' - \vec{r}|} = \frac{1}{[(\vec{r}' - \vec{r}) \cdot (\vec{r}' - \vec{r})]^{1/2}}$$

$$= (|\vec{r}|^2 - 2\vec{r} \cdot \vec{r}' + |\vec{r}'|^2)^{-1/2}$$

$$= \frac{1}{|\vec{r}|} \left(1 - \frac{2\vec{r} \cdot \vec{r}'}{|\vec{r}|^2} + \frac{|\vec{r}'|^2}{|\vec{r}|^2} \right)^{-1/2}$$

$\underbrace{\frac{1}{|\vec{r}|}}_{1/r} \underbrace{\left(1 - \frac{2\vec{r} \cdot \vec{r}'}{|\vec{r}|^2} + \frac{|\vec{r}'|^2}{|\vec{r}|^2} \right)^{-1/2}}_{\text{second order in } |\vec{r}'|/|\vec{r}| \ll 1}$
 $\left((1+\epsilon)^{-1/2} \approx 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 \text{ for } \epsilon \ll 1 \right)$

$$1 - \frac{1}{2} \left(-\frac{2\vec{r} \cdot \vec{r}'}{|\vec{r}|^2} + \frac{|\vec{r}'|^2}{|\vec{r}|^2} \right) + \frac{3}{8} \left(\frac{2\vec{r} \cdot \vec{r}'}{|\vec{r}|^2} \right)^2$$

$$= 1 + \frac{\vec{r} \cdot \vec{r}'}{|\vec{r}|^2} + \frac{1}{2|\vec{r}|^2} \left(3 \frac{(\vec{r} \cdot \vec{r}')^2}{|\vec{r}|^2} - |\vec{r}'|^2 \right)$$

$$\Phi(\vec{r}) = \int_V \rho(\vec{r}') \left[\frac{1}{r} \left(1 + \frac{\vec{r} \cdot \vec{r}'}{|\vec{r}|^2} + \frac{1}{2|\vec{r}|^2} \left(3 \frac{(\vec{r} \cdot \vec{r}')^2}{|\vec{r}|^2} - |\vec{r}'|^2 \right) \right) \right] d^3\vec{r}'$$

$$= \frac{1}{r} \underbrace{\int_V \rho(\vec{r}') d^3\vec{r}'}_{Q \text{ (total charge)}} + \frac{\vec{r}}{r^3} \cdot \underbrace{\int_V \rho(\vec{r}') \vec{r}' d^3\vec{r}'}_{\vec{p} \text{ (dipole moment)}}$$

$$+ \frac{3}{2r^5} \int_V \rho(\vec{r}') \left((\vec{r} \cdot \vec{r}')^2 - \frac{1}{3} |\vec{r}|^2 |\vec{r}'|^2 \right) d^3\vec{r}'$$

$$\left(\vec{r} = (x_1, x_2, x_3) \right. \\ \left. \text{and } \vec{r}' = (x'_1, x'_2, x'_3) \right)$$

$$\begin{aligned} \cdot (\vec{r} \cdot \vec{r}')^2 &= \left(\sum_{i=1}^3 x_i x'_i \right) \cdot \left(\sum_{j=1}^3 x_j x'_j \right) \\ &= \sum_{i=1}^3 \sum_{j=1}^3 x_i x_j x'_i x'_j \\ \cdot |\vec{r}|^2 |\vec{r}'|^2 &= \left(\sum_{i=1}^3 x_i^2 \right) |\vec{r}'|^2 \\ &= \sum_{i=1}^3 \sum_{j=1}^3 x_i x_j (\delta_{ij} |\vec{r}'|^2) \end{aligned}$$

$$\sum_{i=1}^3 \sum_{j=1}^3 x_i x_j \int_V \rho(\vec{r}') \left(x'_i x'_j - \frac{1}{3} \delta_{ij} |\vec{r}'|^2 \right) d^3\vec{r}'$$

$\underbrace{\quad}_{Q_{ij} \text{ (quadrupole moment)}}$

$$\text{So: } \Phi(\vec{r}) \approx \underbrace{\frac{Q}{r}}_{\text{monopole term}} + \underbrace{\frac{\vec{p} \cdot \vec{r}}{r^3}}_{\text{dipole term}} + \underbrace{\frac{3}{2r^5} \sum_{i=1}^3 \sum_{j=1}^3 x_i x_j Q_{ij}}_{\text{quadrupole term}}$$

A variation in time of these terms due to a change of the electric charge distribution results in electromagnetic radiation of decreasing intensity from monopole, to dipole, and quadrupole.

* Monopole: $\dot{Q} = 0$ (charge conservation)

* Dipole: $\dot{\vec{p}} \neq 0$ possible, leading to dipole electromagnetic radiation.

The electromagnetic perturbation is propagated through the vector potential $\vec{A}$.

$$\vec{A} \approx \frac{1}{rc} \int_V \underbrace{\vec{j}(\vec{r}')}_{\text{current density}} d^3\vec{r}' = \frac{\underbrace{\dot{\vec{p}}}_{\text{dipole moment}}}{rc}$$

solution of the propagation equation $\square^2 \vec{A} = \frac{4\pi}{c} \vec{j}$ at the lowest order in the multipole expansion

$$\vec{j} = \rho \vec{v} \quad \text{charge velocity} \quad \vec{v} = \dot{\vec{r}'}$$

$$\underbrace{\frac{dE_{EM}}{dt}}_{\text{Radiated electromagnetic power}} \approx \frac{|\ddot{\vec{p}}|^2}{c^3}$$

For the equivalent mass distribution, the same logic applies but a variation in time results in gravitational wave radiation.

* Monopole: $\dot{M} = 0$ (mass conservation)

* Dipole: $\dot{\vec{P}} = \int_V \underset{\substack{\uparrow \\ \text{mass} \\ \text{density}}}{\rho(\vec{r})} \vec{v} dV = M \underset{\substack{\uparrow \\ \text{center of} \\ \text{mass velocity}}}{\vec{v}_{\text{com}}} = \vec{0} \quad (\text{momentum conservation})$

* Quadrupole: $\ddot{Q}_{ij} = \frac{d^2}{dt^2} \left(\int_V \rho(\vec{r}) \left(x_i x_j - \frac{1}{3} \delta_{ij} |\vec{r}|^2 \right) d^3\vec{r} \right) \neq 0.$

possible, leading to quadrupole gravitational wave radiation.

The space-time perturbation is propagated by h^{ij} .

$h_{ij} \approx \frac{\ddot{Q}_{ij}}{r} \frac{G}{c^4}$

$\square^2 h_{ij} = -\frac{16\pi G}{c^4} T_{ij}$ follows the equation

by analogy with $\vec{A} = \dot{\vec{P}}/rc.$

$P_{\text{GW}} \approx \frac{G}{c^5} \sum_{i=1}^3 \sum_{j=1}^3 |\ddot{Q}_{ij}|^2$

radiated gravitational wave power.

by analogy with $P_{\text{EM}} \approx \frac{|\ddot{\vec{P}}|^2}{c^3}$ (using the $q \rightarrow Gm$ substitution)

to get the correct dimension

Actual equations:

mass quadrupole moment $Q_{ij} = \int_V \rho(\vec{r}) \left(x_i x_j - \frac{1}{3} \delta_{ij} |\vec{r}|^2 \right) d^3\vec{r}.$

$h_{ij} = \frac{2G}{c^4 r} \ddot{Q}_{ij}$

Quadrupole formula for the gravitational wave strain

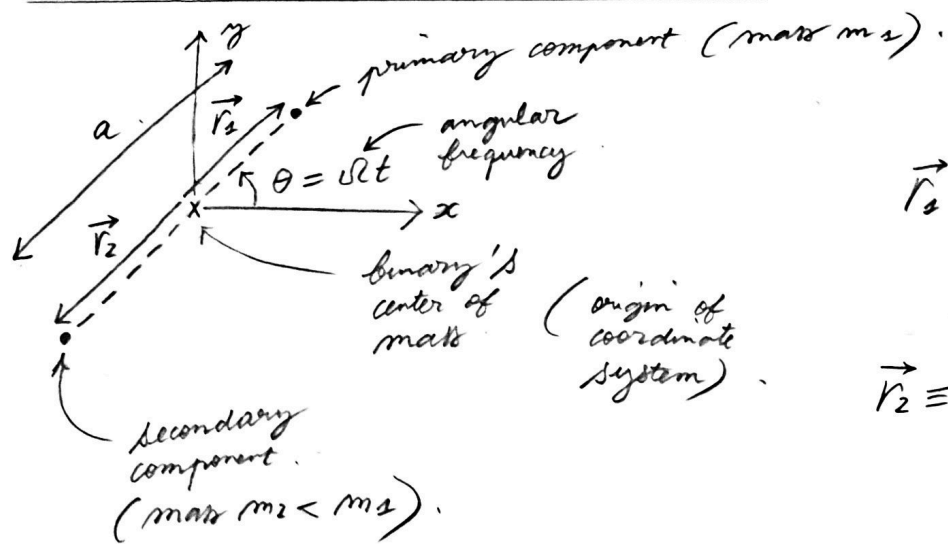
gravitational wave "strain" amplitude

distance to the source.

$P_{\text{GW}} = \frac{G}{5c^5} \sum_{i=1}^3 \sum_{j=1}^3 |\ddot{Q}_{ij}|^2$

Energy per unit time radiated in gravitational waves.

Emission from a binary compact object



$$\vec{r}_1 \equiv \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \frac{|\vec{r}_1|}{r_1} \begin{pmatrix} \cos(\Omega t) \\ \sin(\Omega t) \\ 0 \end{pmatrix}$$

$$\vec{r}_2 \equiv \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \frac{r_2}{|\vec{r}_2|} \begin{pmatrix} -\cos(\Omega t) \\ -\sin(\Omega t) \\ 0 \end{pmatrix}$$

$$Q_{ij} \equiv \int_V \rho(\vec{r}) \left(x_i x_j - \frac{1}{3} r^2 \delta_{ij} \right) d^3 \vec{r} = Q_{ij}^1 + Q_{ij}^2$$

$$Q_{11}^1 = m_1 \left(x_1^2 - \frac{1}{3} r_1^2 \right)$$

$$= m_1 r_1^2 \left(\underbrace{\cos^2(\Omega t) - \frac{1}{3}}_{\frac{1 + \cos(2\Omega t)}{2}} \right)$$

$$= \frac{m_1 r_1^2}{2} \left(\frac{1}{3} + \cos(2\Omega t) \right)$$

$$Q_{12}^1 = m_1 (x_1 y_1)$$

$$= m_1 r_1^2 \underbrace{\cos(\Omega t) \sin(\Omega t)}_{\frac{1}{2} \sin(2\Omega t)}$$

$$= \frac{m_1 r_1^2}{2} \sin(2\Omega t)$$

$$Q_{22}^1 = m_1 \left(y_1^2 - \frac{1}{3} r_1^2 \right)$$

$$= m_1 r_1^2 \left(\underbrace{\sin^2(\Omega t) - \frac{1}{3}}_{\frac{1 - \cos(2\Omega t)}{2}} \right)$$

$$= \frac{m_1 r_1^2}{2} \left(\frac{1}{3} - \cos(2\Omega t) \right)$$

$$Q_{33}^1 = -\frac{1}{3} m_1 r_1^2$$

$$Q_{21}^1 = Q_{12}^1$$

$$Q_{31}^1 = Q_{32}^1 = Q_{13}^1 = Q_{23}^1 = 0$$

Same for Q_{ij}^2 with the substitutions $m_1 \rightarrow m_2$ and $r_1 \rightarrow r_2$.

$$\text{So, } Q_{ij} = \frac{m_1 r_1^2 + m_2 r_2^2}{2} \begin{pmatrix} \frac{1}{3} + \cos(2\Omega t) & \sin(2\Omega t) & 0 \\ \sin(2\Omega t) & \frac{1}{3} - \cos(2\Omega t) & 0 \\ 0 & 0 & -2/3 \end{pmatrix}$$

$$\begin{aligned} & \left\{ \begin{aligned} \cdot r_1 &= \frac{m_2}{M} a \\ \cdot r_2 &= \frac{m_1}{M} a \end{aligned} \right. \quad \begin{aligned} & M \equiv m_1 + m_2 \text{ (total mass)} \\ & a \equiv r_1 + r_2 \text{ (binary separation)} \end{aligned} \\ & \frac{1}{2} \left(m_1 \left(\frac{m_2}{M} a \right)^2 + m_2 \left(\frac{m_1}{M} a \right)^2 \right) \\ &= \frac{1}{2} a^2 \frac{m_1 m_2}{M^2} \underbrace{(m_1 + m_2)}_M \\ &= \frac{1}{2} a^2 \frac{m_1 m_2}{M} = \frac{1}{2} a^2 \mu \quad \mu \equiv \frac{m_1 m_2}{M} \text{ (reduced mass)} \end{aligned}$$

$$\ddot{Q}_{ij} = 2a^2 \mu \Omega^2 \begin{pmatrix} -\cos(2\Omega t) & -\sin(2\Omega t) & 0 \\ -\sin(2\Omega t) & \cos(2\Omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$h_{ij} = \frac{2G}{c^4 r} \ddot{Q}_{ij}$$

↑
quadrupole
formula.

Gravitational waves are emitted with a frequency $f_{GW} = 2f$, that is twice the orbital frequency $f = \frac{\Omega}{2\pi}$ of the binary.

$$\text{So: } \boxed{f_{GW} \approx \frac{1}{\pi} \left(\frac{GM}{a^3} \right)^{1/2}}$$

$$\Omega = \left(\frac{GM}{a^3} \right)^{1/2} \quad (\text{from Kepler's third law})$$

$$h \approx \frac{G}{c^4 r} \ddot{Q} \approx \frac{G a^2 M}{c^4 r} \underbrace{\Omega^2}_{\substack{\text{GM}/a^3 \\ \text{order of magnitude level}}} = \frac{a}{r} \cdot \left(\frac{\overset{R_g}{\widetilde{GM/c^2}}}{a} \right)^2$$

$$\mu = \frac{m_1 m_2}{M} \approx M \quad \begin{array}{l} \uparrow \\ \text{assuming} \\ \text{an equal-} \\ \text{mass binary} \\ (m_1 = m_2 = M/2) \end{array}$$

$$\text{So: } \boxed{h \approx \frac{a}{r} \left(\frac{R_g}{a} \right)^2}$$

$$\begin{cases} f_{GW} \approx 100 \text{ Hz} \\ h \approx 10^{-21} \end{cases} \quad \text{for } \begin{cases} \bullet M = 100 M_\odot \quad (\text{typical binary black hole mass}) \\ \bullet a = 6 R_g \quad (\text{innermost stable circular orbit radius, where the binary is about to merge.}) \\ \bullet r = 1 \text{ Gpc} \quad (\text{typical distance of observation}) \end{cases}$$

Now, we compute the radiated power.

$$\ddot{\ddot{Q}}_{ij} = 4 a^2 \mu \Omega^3 \begin{pmatrix} \sin(2\Omega t) & -\cos(2\Omega t) & 0 \\ -\cos(2\Omega t) & -\sin(2\Omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$P_{GW} = \frac{G}{5c^5} \sum_{i=1}^3 \sum_{j=1}^3 |\ddot{Q}_{ij}|^2 = \frac{32}{5} \frac{G}{c^5} \mu^2 a^4 \Omega^6.$$

$$\parallel (4a^2\mu\Omega^3)^2 \times \left[\underbrace{\cos^2(2\Omega t) + \sin^2(2\Omega t)}_{\parallel 1} + \underbrace{\cos^2(2\Omega t) + \sin^2(2\Omega t)}_{\parallel 1} \right]$$

$$P_{GW} \approx \frac{G}{c^5} \mu^2 a^4 \Omega^6 \approx \frac{GM^2}{c^5} \frac{(GM)^3}{a^5} = \frac{c^5}{G} \left(\frac{R_g}{a} \right)^5$$

order of magnitude level.

$\begin{cases} \mu \approx M \text{ (equal-mass binary)} \\ \Omega^2 = GM/a^3 \end{cases}$

$$So: \quad P_{GW} \approx \frac{c^5}{G} \left(\frac{R_g}{a} \right)^5.$$

$$P_{GW} \approx 10^5 \frac{M \odot c^2}{s}$$

$\uparrow$
 $a = 6R_g$
 (close to merger).

$\uparrow$
 Sun's rest mass energy

$$E_{GW}^{merger} \approx P_{GW} \times \frac{R_g}{c} \approx 100 M \odot c^2$$

$\uparrow$
 Energy emitted during the final phase of merger.

$\uparrow$
 time of the final plunge.

assuming $M = 100 M \odot$

(comparable to the rest mass energy of the binary).

Impact on the binary orbital evolution.

The binary loses orbital energy due to the emission of gravitational waves, causing its separation to decrease until the binary eventually merges.

$$\frac{dE_{orb}}{dt} = -P_{GW}$$

$\nwarrow$ binary's orbital energy $E_{orb} = -\frac{GM\mu}{2a}$

$$\tau_{\text{merger}} \approx \frac{|E_{\text{orb}}|}{P_{\text{GW}}} \approx \frac{GM^2/a}{c^5 (R_g/a)^5 / G} = \frac{GM}{c^2} \frac{GM}{a c^2} \left(\frac{a}{R_g}\right)^5$$

time needed for the binary to merge due to gravitational wave emission.

$$|E_{\text{orb}}| \approx \frac{GM^2}{a}$$

$\mu \approx M$ (equal-mass binary)

So: $\tau_{\text{merger}} \approx \frac{R_g}{c} \left(\frac{a}{R_g}\right)^4$ Gravitational wave merger timescale.

$$\tau_{\text{merger}} \approx 1 \text{ Gyr.}$$

for: $\begin{cases} M = 100 M_\odot \\ a = 0.1 \text{ AU} \approx 10^5 R_g \end{cases}$

Let's look at the frequency evolution.

$$\frac{dE_{\text{orb}}}{dt} = \frac{GM\mu}{2a} \frac{\dot{a}}{a} = \frac{(GM)^{2/3} \mu}{2} \Omega^{2/3} \left(-\frac{2}{3} \frac{\dot{\Omega}}{\Omega}\right) = -\frac{(GM)^{2/3} \mu}{3} \frac{\dot{\Omega}}{\Omega^{1/3}}$$

$$E_{\text{orb}} = -\frac{GM\mu}{2a}$$

$$\Omega = \left(\frac{GM}{a^3}\right)^{1/2} \propto a^{-3/2}$$

so: $\begin{cases} a = \left(\frac{GM}{\Omega^2}\right)^{1/3} \\ \frac{\dot{a}}{a} = -\frac{2}{3} \frac{\dot{\Omega}}{\Omega} \end{cases}$

$$a^3 = \frac{GM}{\Omega^2}$$

$$P_{\text{GW}} = \frac{32}{5} \frac{G}{c^5} \mu^2 a^4 \Omega^6 = \frac{32}{5} \frac{G}{c^5} \mu^2 (GM)^{4/3} \Omega^{10/3}$$

$$\frac{dE_{\text{orb}}}{dt} = -P_{\text{GW}}$$

$$\frac{(GM)^{2/3} \mu}{3} \frac{\dot{\Omega}}{\Omega^{1/3}} = \frac{32}{5} \frac{G}{c^5} \mu^2 (GM)^{4/3} \Omega^{10/3}$$

$$\dot{\Omega} = \frac{96}{5} \frac{G^{5/3}}{c^5} \underbrace{M^{2/3}}_{\substack{\text{III} \\ \mu_c^{5/3} \leftarrow \text{Chirp} \rightarrow \text{mass}}} \mu \Omega^{11/3}$$

$\Omega = \pi f_{\text{GW}}$

$$\mu_c = M^{2/5} \mu^{3/5} = \frac{m_1^{2/5} m_2^{3/5}}{(m_1 + m_2)^{1/5}}$$

$$\text{So: } \boxed{\frac{d f_{\text{GW}}}{dt} = \frac{96}{5} \pi^{8/3} \left(\frac{G \mu_c}{c^3} \right)^{5/3} f_{\text{GW}}^{11/3}}$$

Gravitational wave frequency evolution for a circular binary.

$$\underbrace{\int_{f_{\text{GW}}(t)}^{f_{\text{GW}}(t_m)} \frac{d f_{\text{GW}}}{f_{\text{GW}}^{11/3}}}_{\text{II}} = \frac{96}{5} \pi^{8/3} \left(\frac{G \mu_c}{c^3} \right)^{5/3} \int_t^{t_m} dt$$

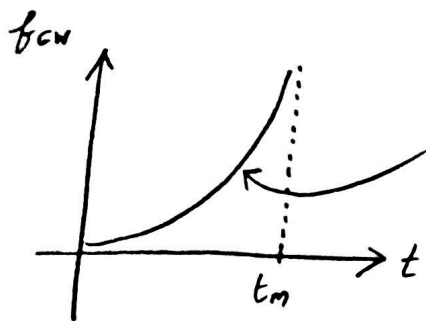
time of binary merger
 $\downarrow$
 t_m
 $\uparrow$ arbitrary time $t < t_m$.

$$\left[-\frac{3/8}{f_{\text{GW}}^{8/3}} \right]_{f_{\text{GW}}(t)}^{f_{\text{GW}}(t_m)} \approx \frac{3}{8 f_{\text{GW}}^{8/3}(t)}$$

III
 $f_{\text{GW}}(t_m) \gg f_{\text{GW}}(t)$

$$\text{So: } f_{\text{GW}} = \frac{5^{3/8}}{8 \pi} \left(\frac{G \mu_c}{c^3} \right)^{-5/8} (t_m - t)^{-3/8}$$

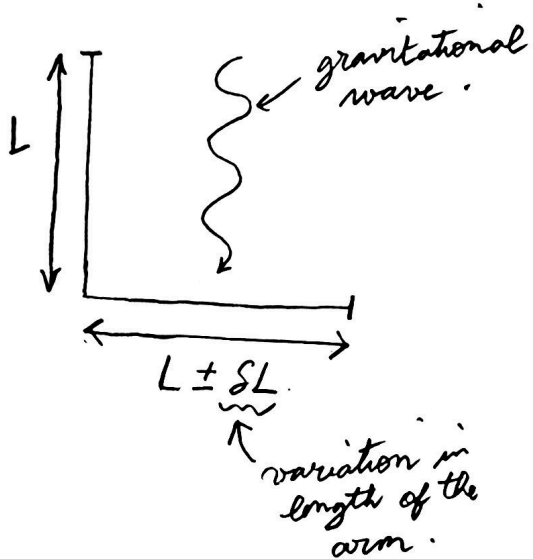
$\uparrow$
 $8 = (256)^{3/8}$



This increase in frequency can be used to estimate the chirp mass from observations.

Detection of gravitational waves

Gravitational waves can be detected through the variation in length of an interferometer's arm that can be very precisely measured using lasers.



The measurement precision needed to detect a gravitational wave is: $\delta L \approx h L \approx 10^{-16} \text{ cm}!$

$$\left\{ \begin{array}{l} \cdot h \approx 10^{-21} \\ \cdot L \approx 1 \text{ km (for the LIGO interferometer)} \end{array} \right.$$