§2 – Observational Cosmology – Cosmological theory discussion

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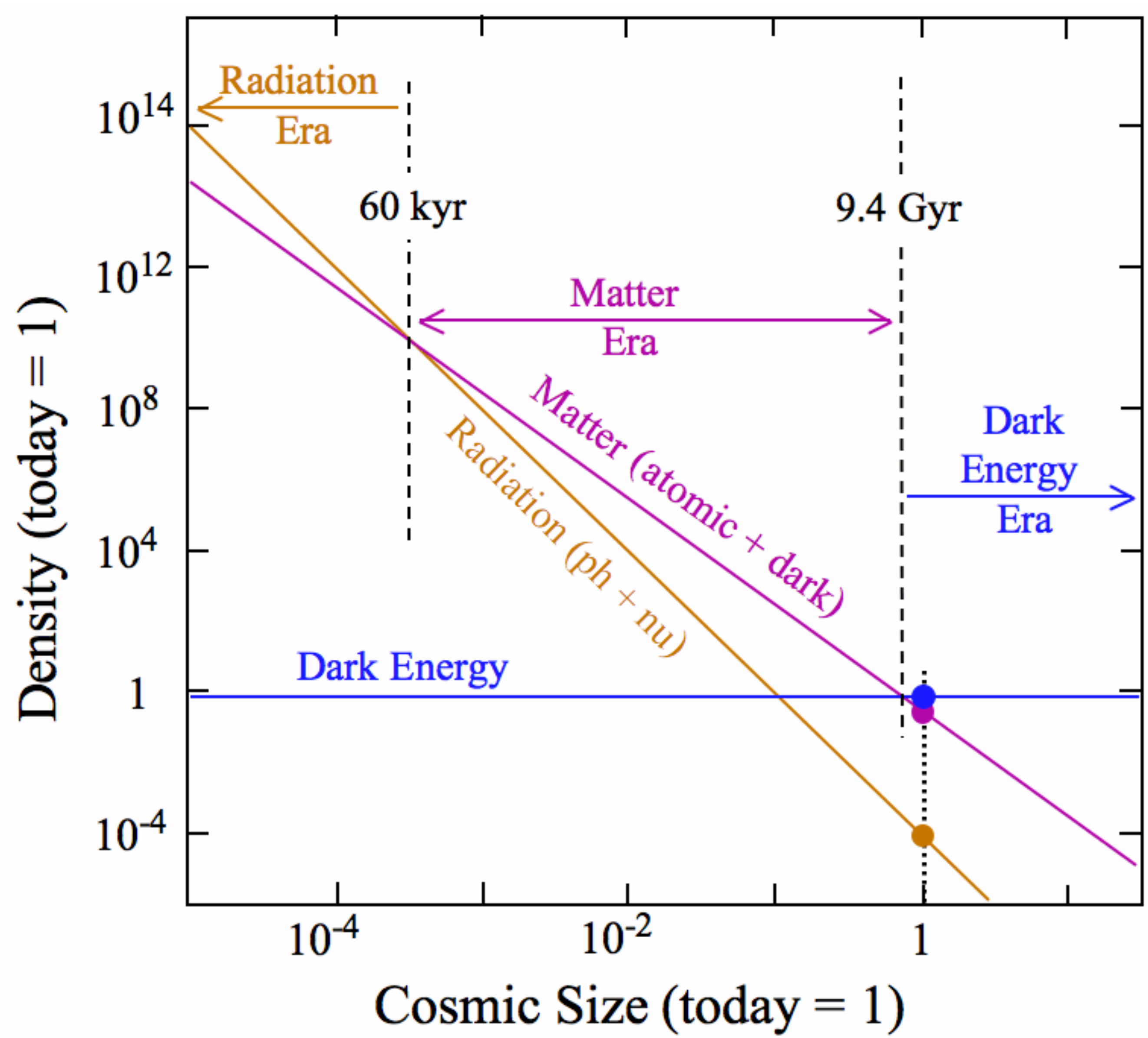
Topic	Sources	Comments
Dynamical equations of the Universe: Assumptions - Newtonian treatment The Friedmann equation The acceleration equation	Course lecture , RR(4.3, 5.1), L1(3.0-3.6) L2(3.0-3.6)	
Solutions to the equations with Lambda=0: For matter (dust); radiation; matter + radiation Flat, open and closed Universes Behaviour of the scale factor with time	L1(4), RR(4.6), unit 2 lecture L2(5)	
Cosmological Parameters : H,q,Omega,Lambda - meanings and definitions	L1(6), RR(4.7), unit 2 lecture L2(6)	
The cosmological constant: Friedmann equation with Lambda term Physical interpretation Dynamical solutions - static & accelerating Universes	L2(7), L1(6.4), RR(4.8), unit 2 lecture (& discussion class)	
Curved space-time Flat, spherical and hyperbolic geometries Relation to the Friedmann models	L1(5), L2(4)	
The Robertson-Walker metric The form and meaning of the R-W metric General Relativity also gives the Friedmann equation	RR(4.4, 4.5), L2(A1.1)	
Cosmological Redshift Relation between redshift and expansion (& scale factor)	L1(4.2), RR(7.4) L2(5.2, A2.1)	
Age of the Universe Dependence on Ho, and Omega (for Lambda=0) Comparison with ages of stars Effect of the cosmological constant	L1(7), RR(4.9 & p.163) L2(8)	

Evolution of the Universe



Evolution of matter, radiation and vacuum contributions

Critical density

- Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{a^2}$$

- Critical density (makes the universe flat, k = 0)

$$\begin{aligned} \rho_\mathrm{c} &= \frac{3H^2}{8\pi G} \\ &= 1.9 \times 10^{-26} h^2 \,\mathrm{kg/m^3} \end{aligned}$$

- Density parameter: $\Omega = \dfrac{\rho}{\rho_c}$

Density parameter

- Density parameter for "species X"

$$\Omega_X = \frac{\varrho_X}{\varrho_c}$$

- Curvature density parameter

$$\Omega_k = -\frac{kc^2}{a_0^2 H_0^2}$$

- (A different form for the) Friedmann equation

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\varrho - \frac{kc^2}{a^2} \\ &= \frac{8\pi G}{3}\varrho_c \Omega - \frac{kc^2}{a^2} \end{aligned}$$

$$\Omega - 1 = -\frac{kc^2}{a^2}$$

$$\Omega + \Omega_k = 1$$

The Einstein Equation(s)

A tensor equation - a shorthand for 16 partial differential eqs., connecting the geometry and mass/energy density:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

Spacetime geometry

Matter distribution

where:

$G_{\mu\nu}$ = The Einstein tensor

$R_{\mu\nu}$ = The Ricci tensor

$g_{\mu\nu}$ = The metric tensor

R = The Ricci scalar

$T_{\mu\nu}$ = The stress-energy tensor

Homogeneity and isotropy requirements reduce this set of 16 eqs. to only 1, $G_{00} = T_{00}$, which becomes the **Friedmann Equation**

Metric and Spacetime

Geometry of space can be generally defined through the **metric**, enabling one to compute the distance between any two points:

$$ds^2 = \sum_a \sum_b g_{ab} dx^a dx^b$$

where g_{ab} is the **metric tensor**. Indices $\{a,b\}$ run 0 to 3, for the spacetime (0 is the time dimension, 1,2,3 are the spatial ones, i.e., xyz)

In a simple Euclidean geometry, it is a diagonal unit tensor (matrix): $g_{aa} = 1$, $g_{a \neq b} = 0$, where $\{a,b\} = \{1,2,3\}$

The metric coefficients g_{ab} are generally functions of the spacetime position, and a proper theory of spacetime has to specify these functions

Metric: Quantifying the Geometry

- The geometry of spacetime is completely specified by a **metric**, $g_{\mu\nu}$
- A special relativistic, Euclidean case is the **Minkowski metric:**

$$ds^2=(c\ dt)^2 - (dx^2 + dy^2 + dz^2)$$

- A general case for a GR, homogeneous and isotropic universe is the **Robertson-Walker metric:**

$$ds^2 = (c\ dt)^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

where $k = -1,0,+1$ for a (negative,flat,positive) curvature

Robertson-Walker Metric

Polar coordinates are useful if all directions are equal (space is *isotropic*):

$$ds^2 = c^2 dt^2 - R^2(t)\left[dr^2 + S_k^2(r)d\varphi^2\right]$$

where

$$S_k(r) = \begin{cases} \sin r, \ (\mathrm{k}=1) & \text{Positive space curvature} \\ \sinh r, \ (\mathrm{k}=-1) & \text{Negative space curvature} \\ r, \quad (\mathrm{k}=0) & \text{Flat (Euclidean) space} \end{cases}$$

If the spatial dimensions expand or contract with time:

$$ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin\theta d\phi^2\right]$$

where $a(t) \equiv \frac{R(t)}{R_0}$ is the **scale factor**

Introducing the Cosmological Constant

Gravitation is an attractive force, so what is to prevent all matter and energy falling to one gigantic lump?

Einstein introduced a negative potential term to balance the attractive gravity:

$$\nabla^2\phi - \lambda\phi = 4\pi G\rho$$

λ could be thought of as an integration constant, or a new constant of nature, or a new aspect of gravity

The Einstein Equations now become:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}$$

This is the **cosmological constant**. Note that the theory does not specify its value, or even the sign!

Essential background cosmology

The evolution of the universe (defined by *a(t)*) is given by the Friedmann equation

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{R_0^2 a^2} \ ,$$

and the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P/c^2) \ ,$$

together with the fluid equation and associated equation of state

$$\dot{\rho} = -\frac{3\dot{a}}{a}(\rho + P/c^2) \ , \quad P = w\rho c^2 \ .$$

Matter or Radiation only solutions

Should be able to get the basic matter- and radiation-dominated scalings from the key equations.

Matter:

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3} \quad ; \quad \rho(t) = \frac{\rho_0}{a^3} = \frac{\rho_0 t_0^2}{t^2} . \qquad H \equiv \frac{\dot{a}}{a} = \frac{2}{3t} ,$$

Radiation:

$$a(t) = \left(\frac{t}{t_0}\right)^{1/2} \quad ; \quad \rho(t) = \frac{\rho_0}{a^4} = \frac{\rho_0 t_0^2}{t^2} .$$

Radiation-dominated universe expands more slowly because of the deceleration supplied by the pressure term.

Matter and radiation mixture solutions

Solutions are often usefully obtained if they are matter or radiation-`dominated’ but have a mixture of both

Matter-dominated:

$$a(t) \propto t^{2/3} \quad ; \quad \rho_{\rm mat} \propto \frac{1}{t^2} \quad ; \quad \rho_{\rm rad} \propto \frac{1}{a^4} \propto \frac{1}{t^{8/3}} \, .$$

Radiation-dominated:

$$a(t) \propto t^{1/2} \quad ; \quad \rho_{\rm rad} \propto \frac{1}{t^2} \quad ; \quad \rho_{\rm mat} \propto \frac{1}{a^3} \propto \frac{1}{t^{3/2}} \, .$$

Notice the radiation in radiation-dominated case falls off more quickly then matter. So, radiation-dominated case is unstable when there is a mixture.

Cosmological constant

Friedmann equation with cosmological constant

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}.$$

Acceleration equation with cosmological constant

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda}{3}.$$

Positive cosmological constant gives a positive contribution to acceleration

Equation of state of cosmological constant

The cosmological constant can be thought of as a 'fluid' with an energy density:

$$\rho_\Lambda \equiv \frac{\Lambda}{8\pi G}$$

The fluid equation becomes:

$$\dot{\rho_\Lambda} + 3\frac{\dot{a}}{a}\left(\rho_\Lambda + \frac{p_\Lambda}{c^2}\right) = 0\,.$$

This must mean:

$$p_\Lambda = -\rho_\Lambda c^2\,.$$

More generally:

$$p_Q = w\rho_Q c^2\,,$$

Age of the Universe

Estimate from the Hubble constant today:

$$H_0 = 100h\,\mathrm{km\,s^{-1}\,Mpc^{-1}}\,,$$

$$H_0^{-1} = 9.77\,h^{-1} \times 10^9\ \mathrm{yrs}\,.$$

Somewhat more accurately:

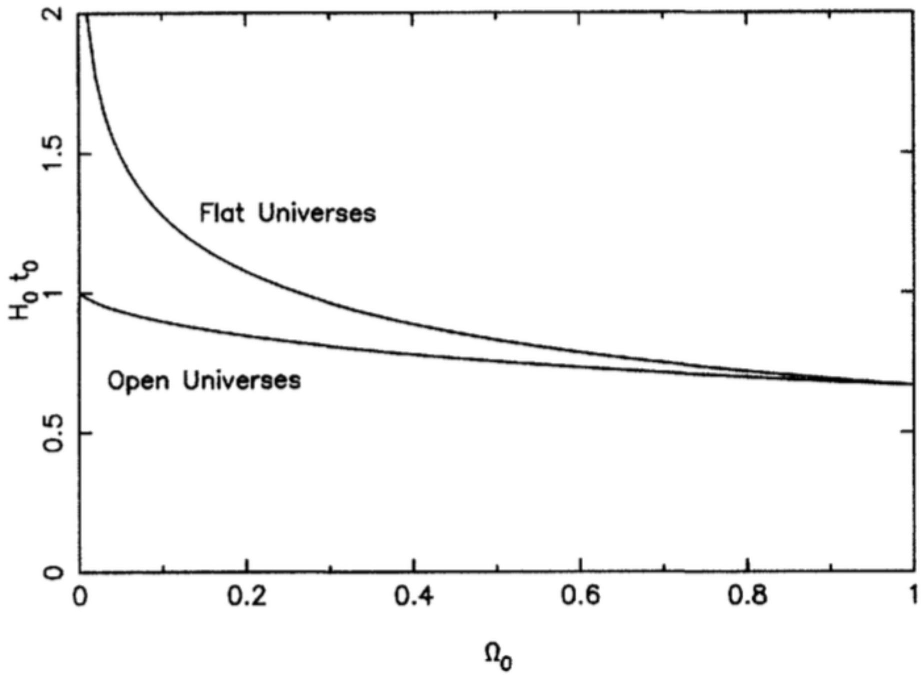
$$a(t) = \left(\frac{t}{t_0}\right)^{2/3} \quad \Longrightarrow \quad H \equiv \frac{\dot{a}}{a} = \frac{2}{3t}\,,$$

$$t_0 = \frac{2}{3}H_0^{-1} = 6.51\,h^{-1} \times 10^9\ \mathrm{yrs}\,.$$

Age of the Universe

More complete solution:

$$H_0 t_0 = \frac{2}{3}\frac{1}{\sqrt{1-\Omega_0}}\ln\left[\frac{1+\sqrt{1-\Omega_0}}{\sqrt{\Omega_0}}\right] = \frac{2}{3}\frac{1}{\sqrt{1-\Omega_0}}\sinh^{-1}\left[\sqrt{\frac{1-\Omega_0}{\Omega_0}}\right]$$



Solutions with a cosmological constant

Now, a ‘closed’ universe (k > 0) doesn’t necessarily recollapse.

$$q_0 = \frac{\Omega_0}{2} - \Omega_\Lambda \, ,$$