Part II
Dynamics in a Gravitational Field

A system of stars behaves like a fluid, but one with unusual properties.

In a normal fluid two-body interactions are crucial to its dynamics, but close encounters between stars are very rare.

Instead the dynamics of a star can be expressed in terms of its interaction with the mean gravitational field of all the other stars (and other matter) in the system.

Definitions

The gravitational field at a point $\mathbf{x}$, defined as the gravitational force on a unit mass, arising from a continuous mass distribution $\rho(\mathbf{x}')$, is given by

$$\mathbf{F}(\mathbf{x}) = G \sum \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \, \rho(\mathbf{x}') \, d^3\mathbf{x}'. \tag{2.1}$$

A more useful quantity is the gravitational potential

$$\Phi(\mathbf{x}) = -G \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|} d^3\mathbf{x}', \tag{2.2}$$

such that $\mathbf{F}(\mathbf{x}) = -\nabla\Phi(\mathbf{x})$.

Circular Velocity (or Circular Speed)

An important quantity, often used in spherically symmetric distributions, is the *circular velocity*, $v_c\,(r)$.

This is the velocity a test particle would have in a circular orbit of radius r about the origin of the mass distribution.

If $M\,(< r)$ be the mass within radius r of a spherical distribution, then

$$\frac{v_c^2(r)}{r} = -F_r(r) = \frac{d\Phi}{dr} = \frac{GM\,(< r)}{r^2}. \tag{2.3}$$

with $F_r(r)$ the radial force from the gravitational potential.

Thus, the circular velocity is a measure of the mass inside of r.

A related quantity is the *escape velocity* $v_{esc}(r)$, which is the velocity required to escape to $r = \infty$.

Equating the kinetic energy with gravitational energy of a test mass:

$$v_{esc}(r) = [2|\Phi\,(r)|]^{1/2}. \tag{2.4}$$

Only when the speed of a star is greater than this value does its (positive) kinetic energy $\frac{1}{2}mv^2$ exceed the absolute value of its (negative) potential energy.

Hence the star can escape from the gravitational field represented by the potential Φ.

Poisson's equation

Poisson's equation is one of the most useful equations of stellar dynamics:

$$\nabla^2\Phi(\mathbf{x}) = 4\pi G\rho(\mathbf{x}). \tag{2.5}$$

It is the gravitational analogue of Gauss's law in electrostatics, and can be derived by taking the divergence of equation (2.1), and applying the divergence theorem.

A derivation is available in most books on dynamics (e.g. Binney & Tremaine §2.1).

In the case of spherically symmetric potentials this takes the form

$$\nabla^2\Phi(r) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right). \tag{2.6}$$

Integrating both sides of equation (2.5) over an arbitrary volume containing mass M, and applying the divergence theorem, we obtain

$$\int \nabla\Phi \cdot d\mathbf{S} = 4\pi GM, \tag{2.7}$$

which can be expressed as:

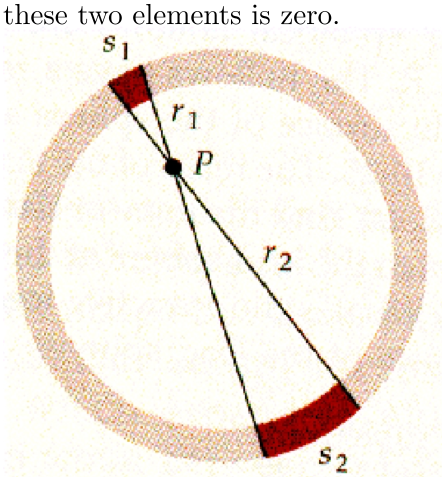
The integral of the normal component of $\nabla\Phi$ over a closed surface is equal to $4\pi G$ times the mass contained within that surface (Gauss's Theorem).

Spherical systems: Newton's Theorems

The most useful results that enable us to calculate the gravitational field and potential of any spherically symmetric distribution of matter are due to Newton.

Newton I: The net gravitational force exerted by a spherical shell of matter on a particle at a point inside the shell is identically zero.

- Consider the cones originating from the point P intersecting the spherical uniform shell of matter at distances r_1 and r_2.
- The circles of intersection have relative areas πr_1^2 and πr_2^2 respectively.
- If the mass per unit area of the shell is σ, it is easy to see that the net gravitational force at P due to these two elements is zero.



This argument can be repeated with cones centred at P that intersect the rest of the sphere, and hence the particle at P experiences no net force from the shell.

Implies that the gravitational potential $\Phi(\mathbf{x})$ inside the sphere is constant, since $\mathbf{F}(\mathbf{x}) = -\nabla\Phi(\mathbf{x})$.

The easiest place to evaluate it is at the centre, which is equidistant from every point on the sphere, which implies

$$\Phi = -\frac{GM}{R}. \tag{2.8}$$

Theorem: Newton II: The gravitational force on a particle that lies outside a closed spherical shell of matter is the same as it would be if all the shell's matter were concentrated into a point at the centre of the shell.

The proof is not easy, but is easily found in textbooks.

These two theorems enable us to calculate the gravitational potential at a radius R, due to an arbitrary spherically symmetric mass distribution of density $\rho(r)$.

Split into two parts: the contribution from shells with $r < R$ and those with $r > R$:

$$\Phi(R) = -4\pi G\left[\frac{1}{R}\int_0^R \rho(r)r^2\,dr + \int_R^\infty \rho(r)r dr\right]. \tag{2.9}$$

Of course, the gravitational force will have contributions only from the shells with $r < R$.

Remember that

$$M(r) = \int_0^r 4\pi\rho(r)r^2\,dr. \tag{2.10}$$

Gravitational Potential Energy

The total gravitational potential energy of a mass distibution is obtained by summing the mutual interaction of all pairs of particles and dividing by two, to avoid double counting. For the continuous case, this gives

$$V = \frac{1}{2}\int \rho(\mathbf{x})\Phi(\mathbf{x})d^3\mathbf{x}. \tag{2.11}$$

A Few Spherically Symmetric Examples

Now we can apply the above results to some simple/useful cases.

Point mass:

For a point mass, the system is analogous to the case of the solar system and is often called the *Keplerian* case:

$$\Phi(r) = -\frac{GM}{r}; \qquad v_c(r) = \left(\frac{GM}{r}\right)^{\frac{1}{2}}; \qquad v_{esc}(r) = \left(\frac{2GM}{r}\right)^{\frac{1}{2}}. \tag{2.12}$$

The circular velocity declines with radius as $v_c \propto r^{-1/2}$, which should be the trend far outside any finite mass distribution.

Homogeneous sphere:

If the density ρ inside a sphere is constant, then

$$M(r) = \frac{4}{3}\pi r^3 \rho, \tag{2.13}$$

and the circular velocity is

$$v_c = \left(\frac{4\pi G\rho}{3}\right)^{\frac{1}{2}} r. \tag{2.14}$$

The circular velocity in this case rises linearly with radius.
This means that the angular velocity $\omega = v/r$ is constant.
The body in question thus moves like a solid body.

Rotation curves of spiral galaxies

When the circular velocity of neutral (HI) hydrogen gas was measured well outside the visible limits of spiral galaxies by radio-astronomers, it was expected that these velocities would decline in a Keplerian fashion with distance from the centre.
Instead, the "rotation curves" of an overwhelming majority of spiral galaxies, representing $v_c(r)$, were found to be almost independent of r out to several times the optical radii of these galaxies.

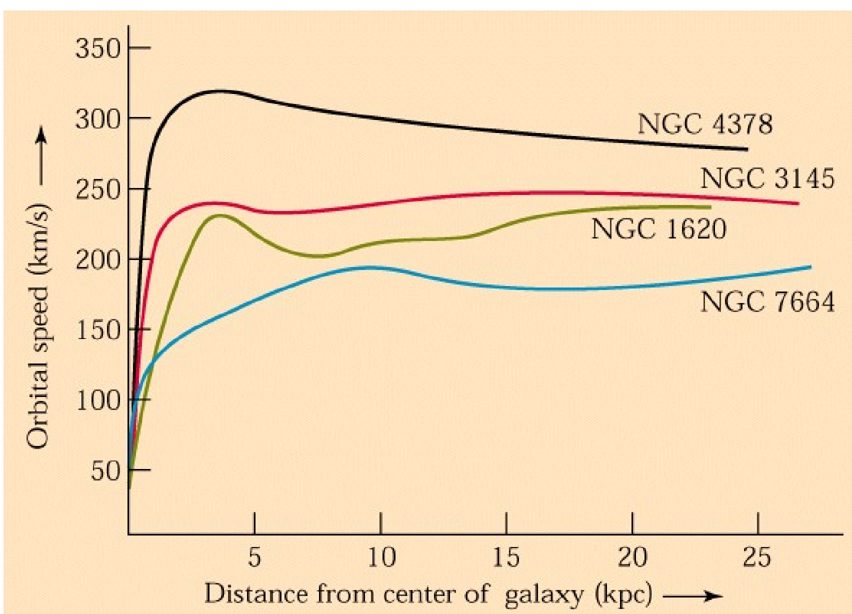


Figure 2.1: Spiral galaxy rotation curves

Power-law density profile:

A spherically symmetric system with a density that falls off as some power of the radius

$$\rho(r) = \rho_0 \left(\frac{r}{r_0}\right)^{-\alpha}, \tag{2.15}$$

is singular at the origin if $\alpha > 0$.
The mass interior to radius r is

$$M(r) = \frac{4\pi\rho_0 r_0^{\alpha}}{3-\alpha} r^{3-\alpha}, \tag{2.16}$$

which means that the mass grows without limit if $\alpha < 3$.
The corresponding circular velocity is given by

$$v_c^2(r) = \frac{GM(r)}{r} = \frac{4\pi G\rho_0 r_0^{\alpha}}{3-\alpha} r^{2-\alpha}. \tag{2.17}$$

Since spiral rotation curves are flat, i.e., v_c=constant, this suggests that within the halos of disk galaxies, the mass density ρ is proportional to r^{-2} (see below).
However, the escape velocity

$$v_{esc}^2(r) = 2\int_r^{\infty} \frac{GM(r)}{r^2} dr = \frac{2}{\alpha - 2} v_c^2(r) \tag{2.18}$$

is finite as long as $\alpha > 2$.
Over the range $3 > \alpha > 2$, the ratio v_{esc}/v_c rises from $\sqrt{2}$ to infinity.

Different Forms of Potential

1. The Hernquist potential

$$\Phi(r) = -\frac{GM}{r+b}; \tag{2.19}$$

2. The Plummer potential

$$\Phi(r) = -\frac{GM}{\sqrt{r^2+b^2}}; \tag{2.20}$$

3. The Jaffe potential

$$\Phi(r) = \frac{GM}{b} \ln\left(\frac{r}{r+b}\right), \tag{2.21}$$

where M and b are constants.

Potential-Density Pairs

Hernquist Potential:

$$\Phi(r) = -\frac{GM}{r+b}, \qquad \rho(r) = \frac{M}{2\pi b^3}\frac{b^4}{r(r+b)^3}, \qquad v_c^2 = GM\frac{r}{(r+b)^2} \tag{2.22}$$

Plummer Potential

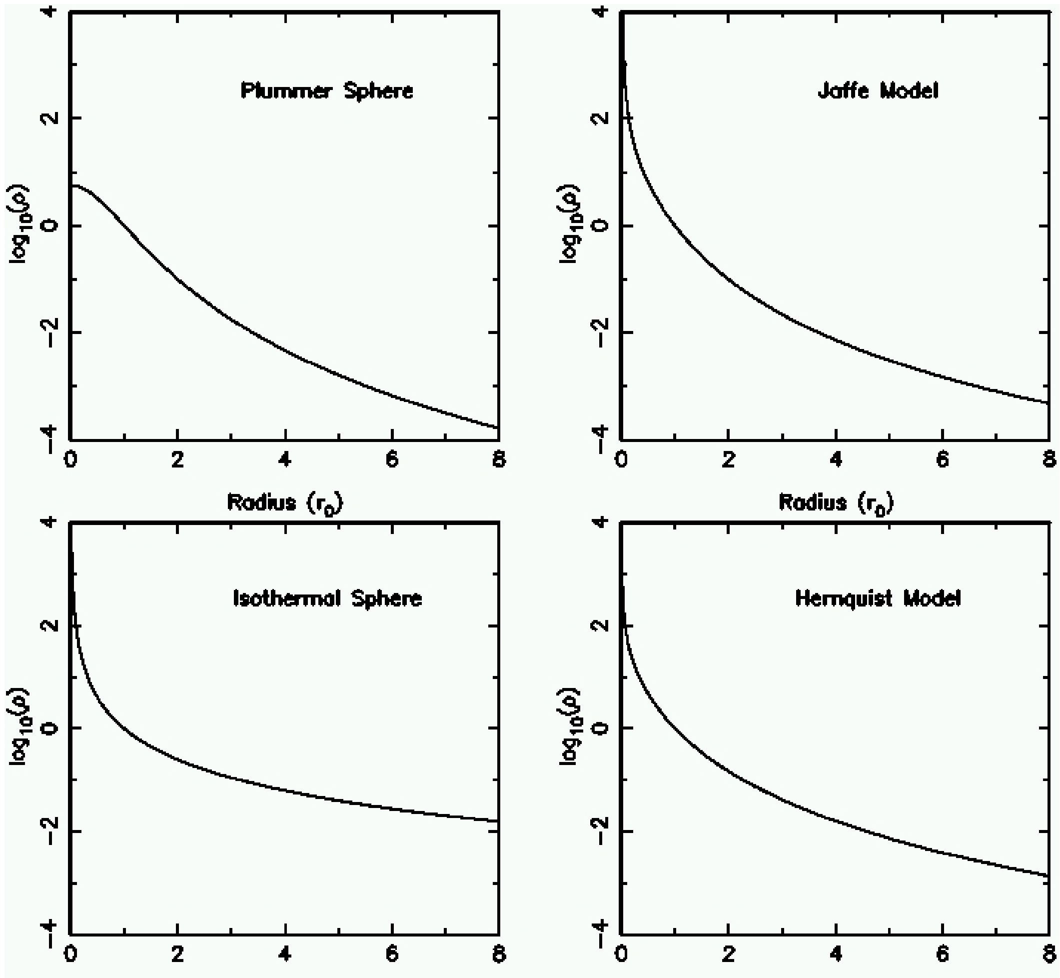


Figure 2.2: Density profiles of common models (from `http://www.astro.utu.fi/~cflynn/galdyn/lecture3.html`).

$$\Phi(r) = -\frac{GM}{\sqrt{r^2+b^2}}, \qquad \rho(r) = \frac{3M}{4\pi b^3}\frac{b^5}{(r^2+b^2)^{5/2}}, \qquad v_c^2 = GM\frac{r^2}{(r^2+b^2)^{3/2}} \tag{2.23}$$

Jaffe Potential

$$\Phi(r) = -\frac{GM}{b}\ln\frac{r+b}{r}, \qquad \rho(r) = \frac{M}{4\pi b^3}\frac{b^4}{r^2(r+b)^2}, \qquad v_c^2 = GM\frac{1}{r+b} \tag{2.24}$$

The Plummer density profile has a finite-density core and the density falls off as r^{-5} as $r \to \infty$, which is a steeper fall-off than is generally seen in galaxies.

The Hernquist and Jaffe profiles both decline like r^{-4} at large r, which has a more sound theoretical basis involving violent relaxation.

The Hernquist model has a gentle power-law cusp at small radii, while the Jaffe model has a steeper cusp.

The Singular Isothermal Sphere

We saw above that the rotation curves of almost all spiral galaxies are remarkably flat away from their centre, instead of being the expected Keplerian form ($v_c \sim r^{-1/2}$).

This means, at large radii, the mass of a spiral galaxy goes as $M(r) \propto r$ and density $\rho \propto r^{-2}$.

This provided early evidence (in the early 1970s) that the outer parts of galaxies have copious amounts of *Dark matter*.

This also means that unless the distribution of matter is cut off at some yet undetermined radius, the mass of each galaxy would diverge.

This model density profile is known as the **singular isothermal sphere**.

Unfortunately, the density of such a model diverges as $r \to 0$.

In many applications, "softened" forms –

$$\rho(r) = \frac{\rho_0}{(1 + r/r_0)^2} \tag{2.25}$$

which have a finite density at the centre (and a "core" of radius r_0) are often used.

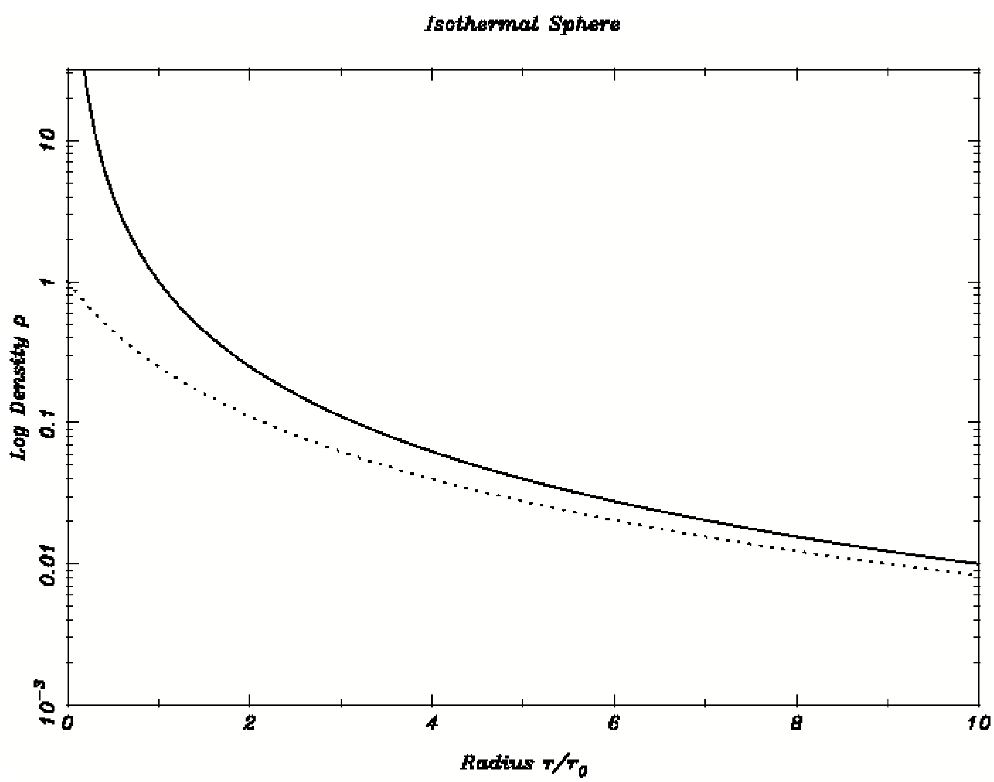


Figure 2.3: The radial dependence of density in the isothermal sphere model. The solid curve represents $\rho \propto r^{-2}$, whereas the dotted curve represents equation (2.25).

The Virial theorem

Before going into details of stellar orbits, it is worth deriving this basic result that applies to the system of gravitating stars as a whole.

In fact it applies to any system of particles bound by an inverse-square force law (e.g. electromagnetism, gravitation), and states that the time-averaged kinetic energy (say $\langle T \rangle$) and the time-averaged potential energy (say $\langle V \rangle$) satisfy

$$2\langle T \rangle + \langle V \rangle = 0. \tag{2.26}$$

Consider the quantity F, defined as:

$$F = \sum_i m_i \dot{x}_i \cdot x_i \tag{2.27}$$

where m_i are the masses of the stars, x_i are the positions, and $\dot{x}_i$ are the velocities.
Clearly

$$\frac{dF}{dt} = 2T + \sum_i m_i \ddot{x}_i \cdot x_i. \tag{2.28}$$

If F is bounded then the long-time average $\langle dF/dt \rangle$ will vanish. Thus

$$2\langle T \rangle + \sum_i m_i \langle \ddot{x}_i \cdot x_i \rangle = 0. \tag{2.29}$$

If the system is gravitationally bound, substituting in for $\ddot{x}_i$, we have

$$2\langle T \rangle - \sum_{ij, i \neq j} G m_i m_j \langle \frac{(x_i - x_j)}{|x_i - x_j|^3} \cdot x_i \rangle = 0. \tag{2.30}$$

Manipulating the indices in the second term and adding, we have

$$2\langle T \rangle - \frac{1}{2} \sum_{i,j,i \neq j} G m_i m_j \langle \frac{1}{|x_i - x_j|} \rangle = 0. \tag{2.31}$$

Remember, that the total potential energy V is:

$$\langle V \rangle = -\frac{1}{2} \sum_{i,j,i \neq j} \frac{G m_i m_j}{|x_i - x_j|} = \frac{1}{2} \sum_i m_i \Phi(x_i) \tag{2.32}$$

The second term in equation (2.31) is now just $-V$, which proves the result.
The Virial theorem provides an easy way to makes rough estimates of masses, because velocity measurements can give $\langle T \rangle$.
It is prudent to consider Virial mass estimates as order-of-magnitude only, because:

1. generally can measure only line-of-sight velocities using redshift measures from spectra, and getting $T = \frac{1}{2} \sum_i m_i \dot{x}_i^2$ from there requires more assumptions (e.g. isotropy of the velocity distribution),

2. the system may not be in a steady state, in which case the Virial theorem does not apply (or applies only approximately).

Clusters of galaxies are particularly likely to be quite far from a steady state (see earlier), in the context of the discussion on crossing times and relaxation times.

Applications of the Virial theorem

Example 1: The Mass of the Perseus cluster of galaxies

The radial velocity dispersion of the Perseus cluster of galaxies is $\sigma_r = 1100$ km/s, and its radius is $2.1\,h_{70}^{-1}$ Mpc, where the Hubble constant is $H_0 = 70\,h_{70}$ km/s/Mpc.

Assuming the cluster to be a sphere of uniform mass density ρ, and applying the Virial theorem $2T + V = 0$, one can work out its mass in the following steps:

Kinetic Energy (T):

Observers measure radial velocities v_r of galaxies from Doppler shifts in their spectra.

The mean of all redshifts in a cluster $\langle v \rangle$ would yield the mean radial velocity of the cluster with respect to the observer (largely due to the Hubble expansion of the Universe).

The dispersion σ_r^2 of the measured values of v_r about this mean would be a measure of the K.E. of the galaxies, so

$$T = 3 \times \frac{1}{2}\,M\sigma_r^2, \tag{2.33}$$

where M is the combined mass of all the galaxies.

The factor 3 accounts for the fact that one measures only the radial component of the velocities of the galaxies, whereas the kinetic energy would depend on the net spatial velocity v of each galaxy, and statistically $\langle v^2 \rangle = 3\langle \sigma_r^2 \rangle$.

Potential Energy (V):

The potential energy of a uniform sphere is calculated from the work done to assemble the sphere out of shells of matter brought in from infinity (work it out yourself!).

Since gravitation is attractive, this quantity would be negative, and is turns out that for a sphere of radius R and mass M, one gets

$$V = -\frac{3}{5}\frac{GM^2}{R}. \tag{2.34}$$

You would get the same answer for the potential energy of a sphere of uniform positive charge due to electrostatic forces, but the sign would be positive.

So, one can estimate the *Virial mass*

$$M = \frac{5\,R\langle \sigma_r^2 \rangle}{G} \tag{2.35}$$

given the radius of the cluster and its radial velocity dispersion.

Answer: Mass of Perseus cluster $M = 3 \times 10^{15}$ $\mathrm{M}_\odot$.

Example 2: The Temperature of the Intergalactic Gas in a Spherical galaxy

Imagine a spherical galaxy forming from a collapsing cloud of gas.

A fraction of the gas has turned into stars, but some is left over in the system as gas.

Assume that the gas is in Virial equilibrium with the gravitational potential of the entire galaxy, dark matter and all.

This interstellar medium (ISM), mostly hydrogen, is an ideal gas.

The mean square velocity of the ISM $\langle v^2 \rangle$ is found by equating

$$\frac{1}{2}\mu m_p \langle v^2 \rangle = \frac{3}{2} k_B T_{vir}, \tag{2.36}$$

where μm_p is the mean mass of each particle of the gas.

If gas is pure atomic hydrogen, then $\mu = 1$.

If gas is ionized hydrogen, then $\mu = 0.5$.

Ionezed cosmic mix then $\mu \sim 0.6$.

k_B is the Boltzmann constant, and T_{vir} is the *Virial temperature.*

Assume that the gas is in dynamical equilibrium with the stars in the galaxy, so if $\langle v^2 \rangle$ is the rms speed of the gas particles of the ISM, then, as in the previous section, $\langle v^2 \rangle = 3\langle v_r^2 \rangle$, where $\langle v_r^2 \rangle$ is the radial velocity dispersion measured from the redshifts of the stars.

From example 1, we have

$$\langle v^2 \rangle = \frac{3GM}{5R} \tag{2.37}$$

Therefore, the Virial temperature of the gas is

$$T_{vir} = \frac{GM}{5R}\frac{\mu m_p}{k_B} = 1 \times 10^6 \mu \left(\frac{M}{10^{11}\, M_\odot} \right) \left(\frac{10\,\mathrm{kpc}}{R} \right) \mathrm{K}. \tag{2.38}$$

Hence, the hot interstellar medium (ISM) of galaxies or the intergalactic (IGM) medium of clusters of galaxies will emit mostly X-rays.

X-ray emitting halos are detected around many elliptical galaxies (and between the galaxies in groups and clusters).

Some tentative evidence of X-ray emitting halos around massive spiral galaxies (still less massive than big ellipticals).