Part III

Stellar Orbits

Introduction to Orbits

Usual approach to modelling galaxies is to look for a way of combining a realistic potential with a distribution of stars following possible orbits within the potential.

Requirement is that the distribution of stars (+dark matter etc) self-consistently provides the mass density distribution that gives rise to the potential we considered in the first place.

The motions of stars (and gas) can tell us where the mass is within galaxies, revealing dark matter and black holes in the centre of galaxies (Sgr A* in our own Galaxy).

Galaxies contain a lot of stars. The gravitational potential within a galaxy consists of a smooth (large scale) component and the deep potential well around individual stars.

We will consider the nature of orbits in potential models of the kind discussed earlier.

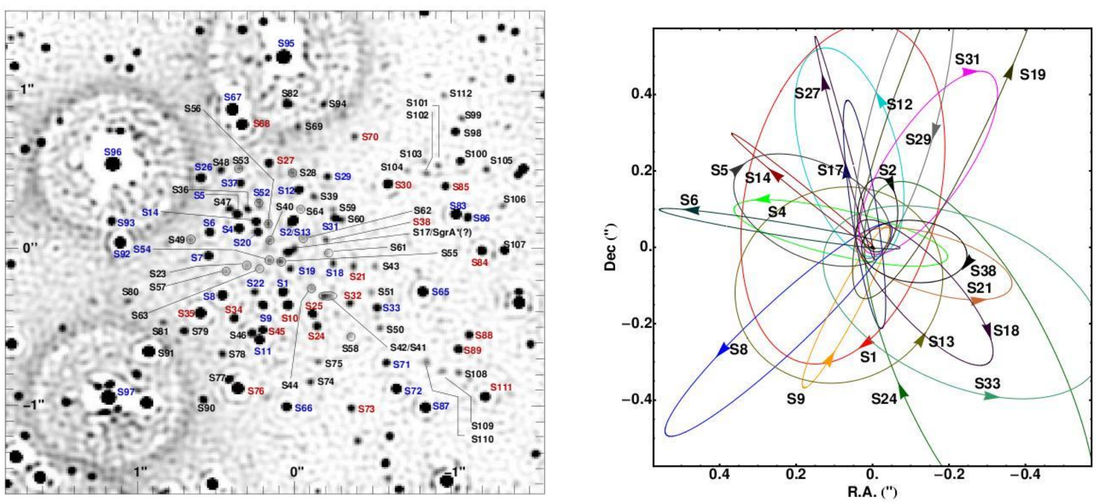

Figure 3.1: Left: A near infrared image of the Galactic Centre region around the central black hole Sgr A*, showing the high density of stars in the central star cluster. Right: By observing this region for many years we can directly see the orbits of the stars. The stars with the most elliptical orbits that pass, close to the black hole, can have velocities of up to ~ 8000 km s^{-1}. The mass of the black hole is estimated to be $\sim 4.3 \times 10^6 M_\odot$, assuming a distance to the Galactic Centre of 8.3 kpc. (Figures from Gillessen et al. 2008).

Integrals of Motion: $I(x, v)$

The motion of any particle can be described by its location in phase space, which in turn is given by the six quantities $\boldsymbol{x}(t), \boldsymbol{v}(t)$.

For example; x, y, z and v_x, v_y, v_z etc

An **Integral of Motion** is a function of phase space coordinates that remains constant along any orbit.

Consider the motion of a star in a *static (time-independent) potential.*

In this case, the energy per unit mass $(\Phi + \frac{1}{2}v^2)$ is an integral of motion.

For a star moving in a *spherically symmetric potential*, the energy and all the components of the angular momentum vector are integrals of motion.

On the other hand, in an *axisymmetric potential*, only the component of the angular momentum along the axis of symmetry is an integral of motion.

Orbital Motion

We will work in spherical $(r, \theta.\phi)$ coordinates.

Consider this plane of motion to be the (r, θ) plane and will ignore any variation in ϕ component.

In spherical polar coordinates, the position $\boldsymbol{r}$ is given by

$$\boldsymbol{r} = r\,\hat{\boldsymbol{r}} + \theta\,\hat{\boldsymbol{\theta}} + \phi\,\hat{\boldsymbol{\phi}} \tag{3.1}$$

The velocity $\boldsymbol{v}$ will be

$$\boldsymbol{v} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\sin\theta\dot{\phi}\hat{\boldsymbol{\phi}} \tag{3.2}$$

Differentiating again, we get the acceleration $\boldsymbol{a}$

$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2)\hat{\boldsymbol{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta} - \frac{1}{2}r\sin 2\theta\dot{\phi}^2)\hat{\boldsymbol{\theta}} + (\ldots)\hat{\boldsymbol{\phi}} \tag{3.3}$$

Remember, we shall only deal with motion in the (r, θ) plane.

As $\dot{\phi} = 0$, in the central force problem the equations of motion are

$$\ddot{r} - r\dot{\theta}^2 = F(r)/m = f(r), \tag{3.4}$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \tag{3.5}$$

where m is the mass of the particle and $F(r)$ the radial force.

Multiplying eqn. (3.5) by r on both sides, and integrating with respect to t, we get the familiar integral of motion

$$r^2\dot{\theta} = \text{constant } (= L/m), \tag{3.6}$$

where L is the (conserved) angular momentum.

From this, it also follows that the area swept out by the line joining the two bodies per unit time, given by $\frac{1}{2}r^2\dot{\theta}$, is conserved (c.f. Kepler's 2nd law).

Note that this result holds for *all* central forces, not just the r^{-2} case, as in the Kepler problem.

Substitute eqn. (3.6) into eqn. (3.4) to yield an equation involving r and its derivatives only:

$$m\ddot{r} - \frac{L^2}{mr^3} = F(r). \tag{3.7}$$

This is the same equation as for a 1-D problem in which a particle of mass m is subject to a force:

$$F'(r) = F(r) + \frac{L^2}{mr^3}. \tag{3.8}$$

The additional term in eqn. (3.8) becomes clearer if written as:

$$\frac{L^2}{mr^3} = mr\dot{\theta}^2 = \frac{mv_\theta^2}{r}, \tag{3.9}$$

the familiar centrifugal force.

The corresponding potential (called the **effective potential**) is given by

$$V'(r) = V(r) + \frac{1}{2}\frac{L^2}{mr^2}. \tag{3.10}$$

Furthermore, the energy conservation relation implies that the total energy

$$E = V'(r) + \frac{1}{2}m\dot{r}^2 = V(r) + \frac{1}{2}\frac{L^2}{mr^2} + \frac{1}{2}m\dot{r}^2 \tag{3.11}$$

is constant.

The Inverse-Square Central Force

For an inverse-square central force, e.g. gravitation due to a central point mass,

$$F(r) = -\frac{k}{r^2}, \qquad V(r) = -\frac{k}{r}, \tag{3.12}$$

and the corresponding effective potential is

$$V'(r) = -\frac{k}{r} + \frac{L^2}{2mr^2}. \tag{3.13}$$

This quantity is plotted in Fig. (3.2) as a function of r, where the two dashed lines represent the first and second terms on the right-hand side respectively, and the solid line is the sum $V'(r)$.

1. Consider the motion of a particle with energy E_1, such that $E_1 \gg 0$.

   If $r < r_1$, then $V' > E_1$, and KE ($\frac{1}{2}m\dot{r}^2$ in eqn. (3.11)) will have to be negative. This is not possible since it involves the square of the velocity.

   This means that a star of energy E_1 can never come closer than r_1 to the centre.

   Since the high positive value of E_1 is due to the angular momentum L, it follows that in a two-body system with substantial angular momentum, neither of the particles can pass through the centre in their orbit.

   For $E_1 > 0$ – hyperbolic orbit. As $r \to \infty$ KE is still positive. Unbound orbit.

2. $E_2 = 0$ we have a parabolic orbit.

3. For a particle with energy $E_3 < 0$, in addition to a lower bound r_2, there is also a maximum value r_3 that cannot be exceeded with positive kinetic energy.

   Stars with this energy will be bounded, their orbit always lying between $r_2 < r < r_3$ (i.e. elliptical orbits).

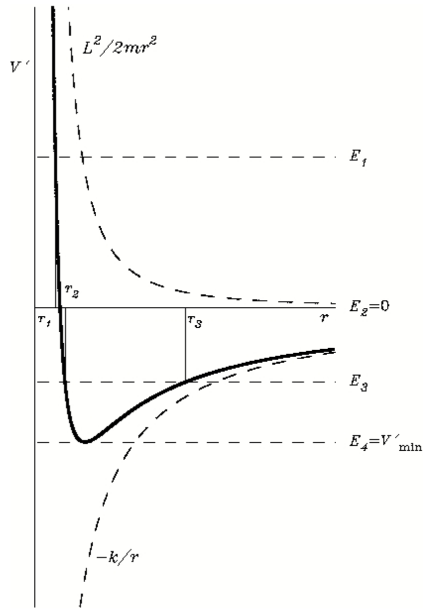

Figure 3.2: The equivalent one-dimensional "effective" potential for an attractive inverse-square central force.

4. Energy E_4 is located at the minimum of the effective potential V'. These two bounds coincide, which means that motion is possible at only one value of r (i.e. circular orbit).

   This occurs when F' is zero, i.e. when $F(r) = -mr\dot{\theta}^2$.

   This is the familiar case where the applied force is equal and opposite to the "reversed effective force" of centripetal acceleration.

5. Finally a star with $E < E_4$ cannot have a feasible orbit, and will free-fall into the centre.

Orbits in Spherically Symmetric Potentials: Examples

The basic set of the equations of motion are eqns. (3.4-3.6).
Writing the specific angular momentum as $\ell = L/m = r^2\dot{\theta} =$ constant, and the acceleration as $f = F(r)/m$, eqn. (3.4) becomes

$$f = \frac{\ell^2}{r^2}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) - \frac{\ell^2}{r^3}. \tag{3.14}$$

Putting $u = 1/r$, one obtains the equation

$$\frac{d^2u}{d\theta^2} + u = -\frac{f}{\ell^2 u^2}. \tag{3.15}$$

Solutions to this equation can be of two types:

1. Unbound orbits, where as time progresses, $u \to 0$ (or $r \to \infty$).

   We are not interested in such orbits, since they do not constitute galaxies.

2. Bound orbits are such that r (and u) oscillate with time between definite bounds.

Multiplying eqn. (3.15) by $du/d\theta$ and integrating, we get the energy equation

$$\left(\frac{du}{d\theta}\right)^2 + \frac{2\Phi}{\ell^2} + u^2 = \frac{2E}{\ell^2} \tag{3.16}$$

where the constant of integration has been written on the right-hand side in terms of the total energy per unit mass E and specific angular momentum ℓ.

Remember that

$$f(r) = -\frac{d\Phi(r)}{dr} = u^2\frac{d\Phi(u)}{du} \tag{3.17}$$

For the bound orbits, at the limits of u, the quantity $du/d\theta$ vanishes, so the two extremes are given by the roots of the quadratic equation

$$u^2 + \frac{2\left[\Phi(u) - E\right]}{\ell^2} = 0, \tag{3.18}$$

between which the star will oscillate.

The inner radius $r_1 = 1/u_1$ is called the *pericentre*, and the outer radius $r_2 = 1/u_2$ the *apocentre*.

Radial and Azimuthal Periods

The energy equation (eqn. 3.16) can be rewritten as

$$E = \Phi + \frac{1}{2}\dot{r}^2 + \frac{1}{2}(r\dot{\theta})^2, \tag{3.19}$$

$\dot{\theta}$ can be eliminated using eqn. (3.6), giving

$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{2(E - \Phi) - \frac{\ell^2}{r^2}}. \tag{3.20}$$

The $\pm$ signs indicate that there are two cases where the star alternately moves towards the centre and away.

From eqn. (3.18) you can verify that $\dot{r} = 0$ at the pericentre and apocentre of the orbit.

The radial period is the time spent in travelling from pericentre to apocentre and back to pericentre,

$$T_r = 2\int_{r_1}^{r_2} \frac{dr}{\sqrt{2(E - \Phi) - \ell^2/r^2}}. \tag{3.21}$$

Likewise the azimuthal period can be defined as the time taken for the star to go through a whole cycle for the other coordinate ($\delta\theta = 2\pi$) in the plane of the orbit.

For a Kepler orbit, for instance, this ratio $T_\theta/T_r = 1$

Consequently, the planet, travelling around the Sun, goes over and over the same elliptical orbit as a result.

The ratio of the two periods T_θ/T_r is in general not a rational number.

Consequently, the typical orbit of a star in a spherically symmetric potential will be a rosette bound between two concentric circles of radius r_1 and r_2.

The star will pass through every point between the two circles, given enough time.

Clearly these orbits, which do not pass through the centre, cannot be the only constituents of galaxies, which do not look like doughnuts.

However, they could easily make up the disks of spiral galaxies. We will look at other kinds of orbits in our discussion of axisymmetric potentials later on.

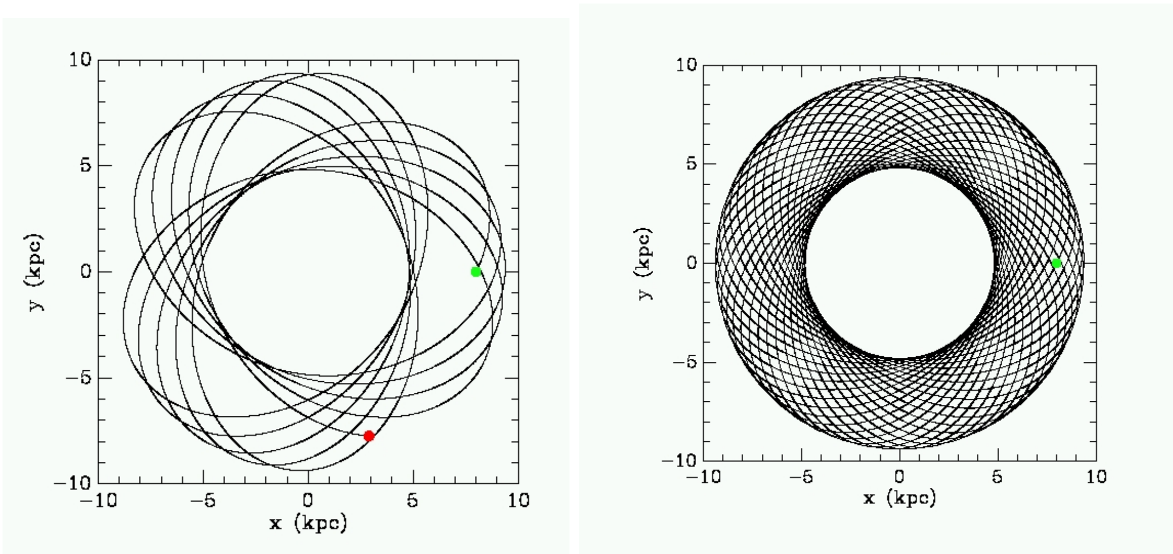

Figure 3.3: The orbit of a star in the potential of the Milky way, given by a spherically symmetric potential that produces a flat rotation curve. A star starts off from the point at $x = 8$ kpc, $y = 0$ kpc, where the Sun is now, with velocity $\boldsymbol{v}$= (90, 180, 0) km/s. Note, that it moves off to the right and top since the x and y components of its initial velocity are positive. The left plot shows its motion over 2 Gyr, and the right one, after 20 Gyr.

The Kepler Orbit

All the mass in the system M is enclosed within the orbit of the star.

The star is a distance r from the centre of attraction.

The potential is spherically symmetric (the same case as for a planet in the Solar system).

The force per unit mass on the star is $f(r) = -GM/r^2 = -GMu^2$.

Consequently, eqn. (3.15) can be rewritten as

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{\ell^2}. \tag{3.22}$$

Since $u = 1/r$, the solution can be written as,

$$r(\theta) = \frac{a(1-e^2)}{1 + e\cos(\theta - \theta_0)}, \tag{3.23}$$

where the eccentricity of this conic section orbit is $e = C\ell^2/GM$.

C being the constant of integration from eqn. (3.22), and the semi-major axis $a = \ell^2/GM(1-e^2)$.

For bound orbits, which we are interested in here, the eccentricity $e < 1$, and r is finite for all values of the azimuthal angle θ, and is a periodic function in 2π.

These orbits are ellipses with the centre of attraction at one of the foci of the ellipse.

The pericentre and apocentre lie at $r_1 = a(1-e)$ and $r_2 = a(1+e)$.

Most often, instead of expressing the radius r as a function the angle θ, you would want to know how it behaves with time t.

The bad news is that in general this cannot be written down in closed form in a single equation.

This is usually represented as a set of parametric equations, in terms of an angular parameter η,

$$r = a(1 - e\cos\eta); \qquad t = \frac{T_r}{2\pi}(\eta - e\sin\eta). \tag{3.24}$$

The radial and azimuthal periods, as mentioned above, are equal in this case.

$$T_r = T_\theta = 2\pi\sqrt{\frac{a^3}{GM}}. \tag{3.25}$$

The Spherical Harmonic Oscillator

Consider the spherically symmetric potential

$$\Phi(r) = \frac{1}{2}\omega^2 r^2 + \text{constant}. \tag{3.26}$$

The corresponding central force is $f(r) = -\omega^2 r$,

We encountered something similar in the case of a sphere of constant density, where the circular velocity $v_c^2(r) \propto r^2$.

We need not derive the equation for u in this case.

Simply writing down the solution in Cartesian coordinates (which is possible in this case) would help us see clearly what is going on.

In Cartesian coordinates, $x = r\cos\theta$ and $y = r\sin\theta$, and the equations of motion in these two directions reduce to $\ddot{x} = -\omega^2 x$ and $\ddot{y} = -\omega^2 y$, where $\omega^2 = 4\pi G\rho/3$.

These are equations of simple harmonic motion.

The solutions are

$$x = X\cos(\omega t + \epsilon_1); \quad y = Y\cos(\omega t + \epsilon_2). \tag{3.27}$$

This orbit is elliptical in general (circular if $X = Y$), with the centre of attraction at the centre of the ellipse.

This is one of two cases that yield closed bounded orbits (Bertrand's theorem). The other is the inverse square case.