# *9. A Cosmological View*

## *9.1 A little cosmology*

From observational evidence, we know the universe is expanding.

- Hubble's Law tells us that the recessional speed of galaxies is proportional to their distance from us.

Here, we aim to develop a simple understanding of how this expansion varies with time.

We begin with a simple Newtonian description: this will suffice in order to allow us to develop definitions for some 'cosmological' quantities.

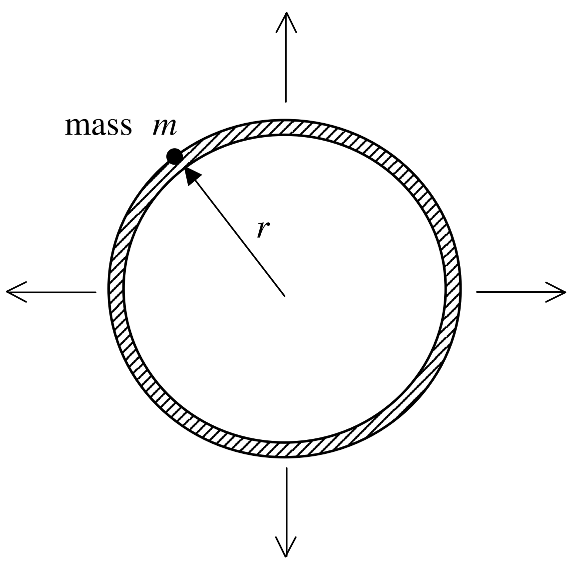


**Figure 9.01: A 'galaxy' on the surface of an expanding sphere**

Consider Figure 9.01. It shows a small mass *m* on the surface of an expanding sphere of radius *r*.

Let us think of this as being a galaxy on the surface of a sphere in the expanding universe.

The mass contained in the thin shell on which our mass *m* lies expands along with the universe: the mass contained within the sphere is therefore conserved.

- The radius *r* is a coordinate distance: it changes with time as the sphere expands, so strictly speaking we should write it as $r(t)$.

- Now, let us 'tag' the location of the surface of the expanding shell with a co-moving coordinate $\sigma$.

- Next, we introduce a scale factor, $R(t)$. At any given time $t$, any thin shell that we define within our expanding 'universe' (e.g., we could consider a second shell *within* the shell shown in Figure 9.01) will have the same value of $R(t)$. It therefore provides us with a time-dependent (but position invariant) measure of the expansion.

- A complete, time-dependent coordinate for the mass is then given by $r(t) = \sigma R(t)$.

- The mass contained within the radius $r$ is just proportional to $r^3\rho$, where here we take $\rho$ to be the mean density of the matter contained within the sphere. For a *specific* shell, this will remain constant over time.

- This is just a statement of the conservation of matter.

- For all shells, $R^3(t)\,\rho(t)$ remains constant over time.

The total energy of the galaxy is conserved during the expansion. It is given by:

$$E = U + \Omega = \frac{1}{2}mv^2 - \frac{GmM}{r} = \frac{1}{2}m\left[\sigma \frac{dR(t)}{dt}\right]^2 - \frac{GmM}{[\sigma R(t)]},$$

(R9.01)

where $M$ is the total mass contained within the expanding sphere.

There are three conditions to consider:

1. If $E < 0$, our ‘universe’ is bounded or closed, and the expansion will eventually come to a halt and be followed by contraction.

2. If $E > 0$, the universe is unbounded or open, and the expansion will continue indefinitely.

3. If $E = 0$ the universe is flat, and the rate of expansion will asymptote to zero at an infinite time; the universe will therefore be infinitely dispersed.

The value of the scale factor at the current epoch, $t_0$, is by definition unity, i.e., $R(t_0) = 1$.

### *9.1.1 The cosmological redshift*

The red shift, $z$, is defined according to:

$$z = \frac{\delta\lambda}{\lambda} . \qquad \text{(R9.02)}$$

- The spectrum formed by bodies (e.g., galaxies) receding from us will be red shifted.

- But remember also that the length scale of the universe, $R(t)$, changes with time.

- So wavelengths will be 'stretched' as the universe enlarges.

- So when we look at very distant objects (on a cosmological scale), whose radiation was emitted at an earlier epoch when $R(t)$ was smaller, the spectrum will be 'stretched' in wavelength by the time we receive it, i.e., it will be cosmologically red shifted.

$$z = \frac{R(t_0) - R(t)}{R(t)} = \frac{R(t_0)}{R(t)} - 1 . \qquad \text{(D9.03)}$$

## 9.2 Formation of structure in the early universe

Next, we consider the conditions in the early universe that led to the formation of observable structure.

### 9.2.1 The density of matter and radiation

First, we investigate how the mass density of matter and radiation changed as the universe expanded after the Big Bang, i.e., as the length scale $R(t)$ increased.

- The mass density varies as:

$$\rho_{\mathrm{m}} \propto R(t)^{-3} . \qquad (9.04)$$

- To get the equivalent radiation density, we recall that the energy density (energy $E$ divided by volume $V$) of radiation is just:

$$\frac{E}{V} = aT^{4} , \qquad (1.06)$$

where $a$ is the radiation constant. Since $E = mc^2$, we have:

$$\rho_{\mathrm{r}} = \frac{m}{V} = \frac{E}{c^{2}} \cdot \frac{aT^{4}}{V} = \frac{aT^{4}}{c^{2}} . \qquad (9.05)$$

- We now recall Wien's Displacement Law for the wavelength at which the emission given by a Black Body is a maximum, i.e., $\lambda_{max}T$ = constant.

- The wavelength $\lambda_{max}$ will be stretched by the increase in $R(t)$. So:

$$\lambda_{\mathbf{max}} \propto R(t),$$

and then:

$$T \propto R(t)^{-1}.$$

- From Equation 9.05, it follows that:

$$\rho_{\mathrm{r}} \propto R(t)^{-4}. \qquad (9.06)$$

- So, over time, $\rho_{\mathrm{r}}$ decreases faster than $\rho_{\mathrm{m}}$ (see Figure 9.02).

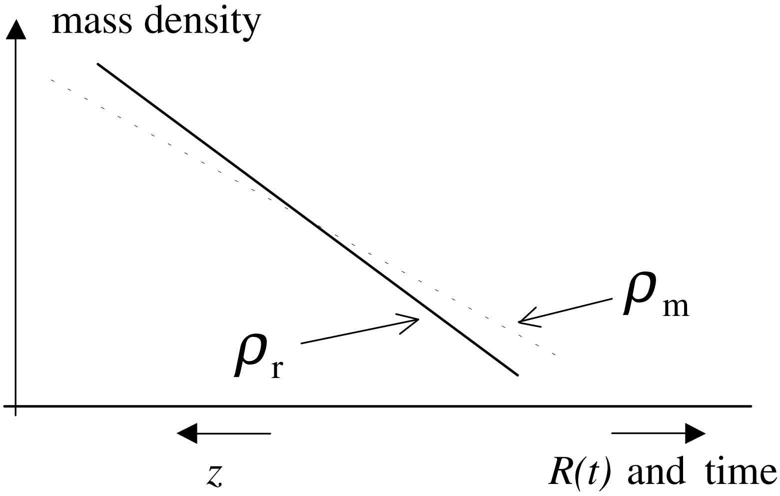


**Figure 9.02: The relative variation of the matter and radiation densities in the universe**

What are the current values for both densities at the present epoch (where by definition, $R(t_0) = 1$)?

- For $T(t_0) = 3$ K, we have $\rho_r = 6.5 \times 10^{-34}$ g cm$^{-3}$.
- The mass density estimated to reside in luminous matter is of the order of $\rho_m = 5 \times 10^{-31}$ g cm$^{-3}$.
- So the current universe is matter dominated.

### *9.2.2 The epoch of recombination*

So, if the universe is now matter dominated, when in the past were the matter and radiation densities similar? (Prior to this the universe will have been radiation dominated.)

From Equations 9.04 and 9.06:

$$\frac{\rho_r}{\rho_m} \propto R(t)^{-1}.$$

- Since the current ratio is $\approx 10^{-3}$, the two densities must have been equal when the scale factor $R(t)$ was $\approx 10^3$ smaller than it is today.

- Remember that the current value, $R(t_0)$, is 1 by definition, so the densities were similar when $R(t) \approx 10^{-3}$.

- This must have been when the red shift $z \approx 10^3$ (cf. Equation 9.03).

What was the temperature at this epoch?

Recall from above that:

$$T \propto R(t)^{-1}.$$

So:

$$\frac{T(t)}{T(t_0)} = \frac{R(t_0)}{R(t)}.$$

With $T(t_0) = 3$ K, we have $T(t) \approx 3000$ K when the densities were similar.

- This temperature is approximately equal to the value above which hydrogen is completely ionised.

- So, after this epoch, electrons and protons combine to form neutral hydrogen.

Now:

- Plasma is much more opaque to radiation than is neutral hydrogen.

- So, prior to recombination, matter interacted far more strongly with radiation than in the current, matter-dominated universe.

- This has important implications for the formation of density fluctuations in the young universe

### *9.2.3 Structure formation*

- To form structures of the type observed in the universe, we require density fluctuations.

- The Jean's condition (cf. Section 8.3) then determines whether or not these can give rise to a collapse of matter to form structures.

Prior to recombination, matter and radiation interact strongly:

- A density perturbation that condenses matter locally will have a similar effect upon the radiation field: this is an adiabatic fluctuation.

- These adiabatic fluctuations tend to be damped out by the flow of radiation from the compressed to the uncompressed regions, i.e., the radiation field tends to try to remain 'smooth'.

- Now, the motion of matter is inhibited by its strong interaction with the radiation: so if density fluctuations in the radiation field cannot grow, they will also be unable to grow in the distribution of matter.

- Matter cannot be compressed independently of the radiation field, so isothermal fluctuations cannot form.

- In an isothermal fluctuation, matter is compressed but the radiation field is not, so that the temperature remains unchanged.

- Recall our discussion in Section 4.3.1 of the conditions that lead to the collapse of a cloud of matter.

    - Ø The Virial Theorem tells us that the gravitational (collapse) term must win out over the kinetic (support) term in the Virial equation in order to initiate collapse.

    - Ø The cloud needs to collapse in an isothermal manner for this condition to be satisfied. Only when it becomes dense and opaque does the cloud behave adiabatically, and this halts the collapse.

- So, after the de-coupling of matter and radiation (i.e., after recombination) isothermal fluctuations can form.

This means that enhancements of density in matter can grow, allowing the formation of gravitationally bound structures.

At the epoch of recombination, what is the Jean's Mass?

Recall that the mass required to initiate spontaneous collapse (the Jeans Mass) is given by

$$M_{\mathrm{J}} = \left( \frac{5kT}{G\mu m_H} \right)^{3/2} \cdot \left( \frac{3}{4\pi \rho_0} \right)^{1/2}, \qquad (8.03)$$

where $\mu$ is the mean weight per particle and $m_H$ is the mass of the hydrogen atom, $T$ is its temperature, and $\rho_0$ is the initial density of the cloud.

We know $T$ at recombination, but what about $\rho_0$? Recall Equation 9.04:

$$\rho_{\mathrm{m}} \propto R(t)^{-3}. \qquad (9.04)$$

We then have:

$$\frac{\rho_{m0}}{\rho_m} = \left( \frac{R(t)}{R(t_0)} \right)^3 ,$$

where $\rho_{\mathrm{m0}}$ is the current mass density. The density at recombination, when $R(t)$ was roughly $10^{-3}$, is therefore about $10^9$ times larger than the current value.

With $\rho_{\mathrm{m0}} = 3 \times 10^{-31}$ g cm$^{-3}$, we have $M_{\mathrm{J}} \approx 10^6\, M_{/}$ .

- This corresponds to the mass of globular clusters.

- Globular clusters contain very old stellar populations, and so our analysis suggests that they were the first large structures to form.

## *9.3 Some observational cosmology*

### *9.3.1 The Hubble constant and the deceleration parameter*

The Hubble constant provides a measure of the rate of expansion of the universe.

A galaxy at a distance $r(t)$ will have a recessional velocity given by:

$$v(t) = H(t)r(t), \qquad (9.07)$$

where $H(t)$ is the Hubble constant. Since the expansion does not occur at a constant rate, $H(t)$ varies with time.

If we denote the current time (or epoch) by $t_0$, then:

- The current value of the Hubble constant is given by

$$H_0 = \frac{1}{R(t_0)} \cdot \frac{dR(t_0)}{dt} = \frac{\dot{R}(t_0)}{R(t_0)}. \qquad \text{(R9.08)}$$

- The dimensionless deceleration parameter, $q$, provides a measure of the rate of expansion of the universe.

As its name implies, if $q > 0$, the universe is decelerating at the epoch when $q$ is determined.

The current value is given by:

$$q_0 = -\ddot{R}(t_0) \cdot \frac{R(t_0)}{\dot{R}^2(t_0)} = -\ddot{R}(t_0) \cdot \frac{1}{R(t_0)H(t_0)} \cdot \qquad (9.09)$$

### *9.3.2 The critical density*

If we set $E = 0$ in Equation 9.01, we can then derive the critical density above which the universe will be bound and closed. A flat universe therefore has this density.

We have:

$$\frac{1}{2} m \left[ \sigma \frac{dR(t)}{dt} \right]^2 = \frac{GmM}{[\sigma R(t)]} \cdot$$

The mass contained within the sphere of radius $\sigma R(t)$ is just:

$$M = \frac{4}{3} \rho \pi [\sigma R(t)]^3 .$$

If we substitute into the equality and reorganise to make density the subject of the equation we have:

$$\rho_c = \frac{3}{8\pi G} \cdot \frac{\dot{R}^2(t)}{R^2(t)} .$$

We can now use Equation 9.08 to introduce the current value of the Hubble constant, $H_0$, so that the equation is now pertinent to conditions at the current epoch, i.e.,

$$\rho_{c0} = \frac{3}{8\pi G} \cdot H_0^2 . \qquad \text{(D9.10)}$$

Equation 9.10 tells us that if the density at the current epoch, $\rho_0$, is equal to the critical density $\rho_{c0}$, the universe will be marginally bound or flat.

The ratio of the observed and critical densities is usually denoted by:

$$\Omega_m = \frac{\rho_0}{\rho_{c0}} = \frac{8\pi G \rho_0}{3H_0^2} , \qquad \text{(D9.11)}$$

where the '*m*' denotes that this is the observed fraction of the critical mass density.

### *9.3.3 Life histories of simple models of the universe*

We now use the arguments and simple equations from above to detail the life history a few simple universes.

Figure 9.03 shows the expansion of a universe that has $\rho > 0$.

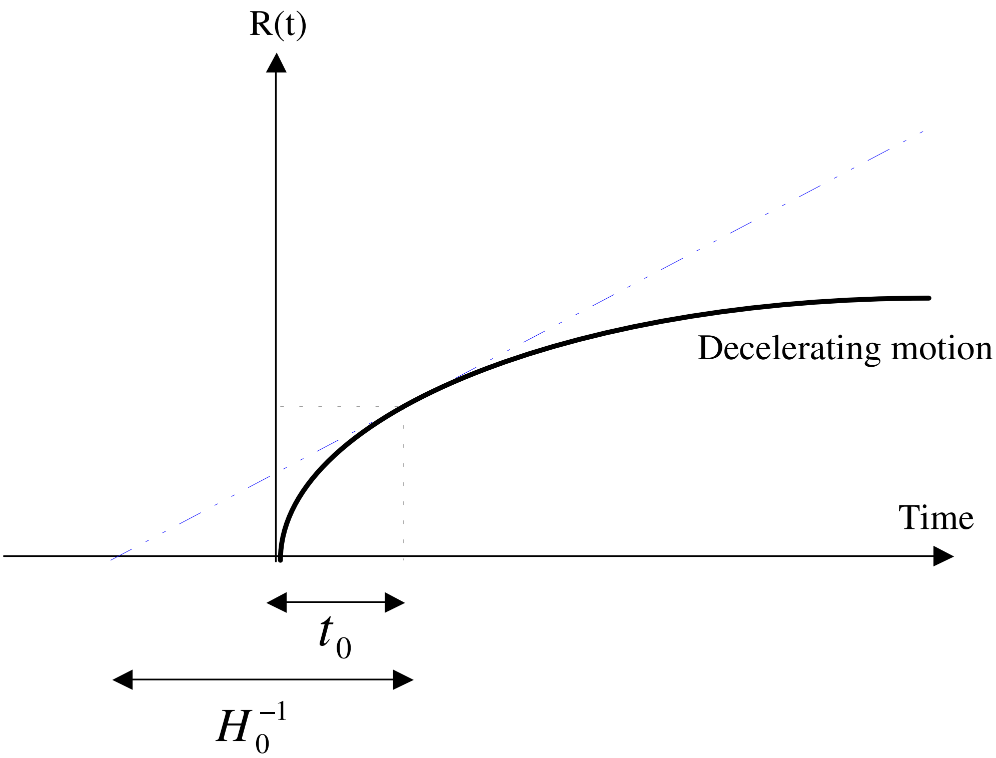


**Figure 9.03: The expansion of the universe**

- As time passes, the rate of expansion decreases.

- The pictorial representation assumes that the life history of this universe can be traced back to a Big Bang at $t = 0$. At that time, the distance scale $R(t) = 0$.

- At the current age of the universe, $t_0$, the scale factor is $R(t_0)$ and the Hubble constant is $H_0$.

- By definition, $R(t_0) = 1$.

- The gradient of the curve at the current epoch, $t_0$, gives the speed of recession $dR(t_0) / dt$.

- Since the expanding universe decelerates, its current age

$$t_0 < H_0^{-1} . \qquad \text{(R9.12)}$$

For the case of a marginally bound (flat) universe with $E = 0$, we find that:

$$t_0^{\text{flat}} = \frac{2}{3} H_0^{-1} . \qquad (9.13)$$

(See non-assessed ***Problem Sheet 3***.)

Next, we consider how the deceleration parameter varies for different universes (Figure 9.04)

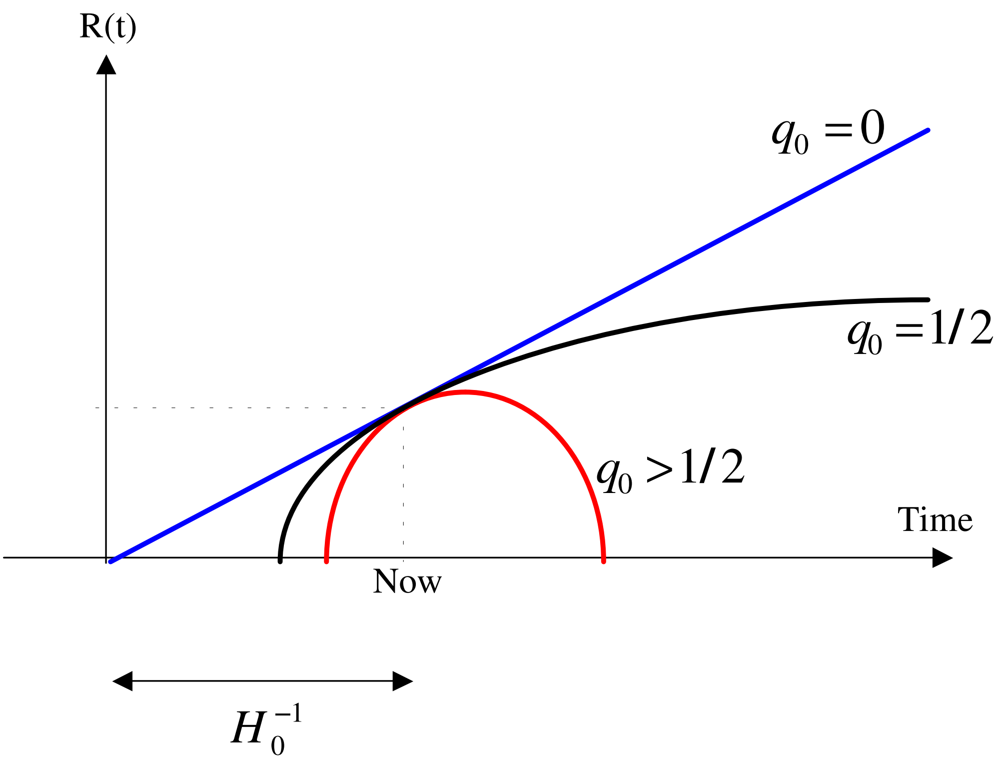


**Figure 9.04: Histories of different universes**

With reference to the figure:

- It can be shown readily that:

$$q_0 = \frac{1}{2}\Omega_m . \qquad (9.14)$$

A flat universe (with critical density, i.e., $E = 0$; $\Omega_m = 1$) therefore has $q_0 = ½$.

- The case with $q_0 = 0$ is just that of an empty ($\rho = 0$) universe.

- For $0 < q_0 < ½$ , we have an unbounded universe ($E > 0$; $\Omega_m < 1$) that will expand forever

- For $q_0 > ½$, the universe is bounded ($E < 0$; $\Omega_m > 1$) and will eventually collapse.

Whichever of the above (or other more complicated models) matches reality, all the models are constrained such that they must give the current value of the Hubble constant, $H_0$.

- This fixes the gradient of the expansion curve at the current epoch.

- In turn, this means that different types of universe that match this constraint will have different ages (see figure again).

### *9.3.4 The cosmological constant*

Consider the equation of motion of a ‘galaxy’ of mass *m* in the universe.

It is attracted toward the centre of the sphere by the mass contained within its radius, i.e.,

$$m\frac{d^2r}{dt^2} = m\sigma\ddot{R}(t) = -\frac{GmM}{\left[\sigma R(t)\right]^2} = -\frac{Gm(4\pi/3)\rho(t)\left[\sigma R(t)\right]^3}{\left[\sigma R(t)\right]^2},$$

So that:

$$\ddot{R}(t) = -\frac{4\pi}{3}\cdot G\rho(t)R(t).$$

A more detailed treatment (i.e., beyond Newtonian mechanics) gives the following equation of motion:

$$\ddot{R}(t) = -\frac{4\pi}{3}\cdot G\rho(t)R(t) + R(t)\cdot\Lambda/3. \qquad (9.15)$$

The additional term, Λ, is called the cosmological constant.

It has units of $s^{-2}$.

We now have two terms on the right-hand side of the equation of motion:

- The first represents the gravitational effect of matter, and this acts as a deceleration term.

- The second can either be positive or negative, depending upon the value of $\Lambda$. If $\Lambda$ is positive, then it can act as a cosmic repulsion term.

Its presence in the equation of motion implies the presence of energy in a universe devoid of matter, i.e., it can be regarded as being the energy density of a vacuum.

The Cosmological Constant can be thought of as giving rise to an equivalent density, $\Omega_\Lambda$, such that:

$$\Omega_\Lambda = \frac{\Lambda}{3H_0^2}. \qquad (9.16)$$

Figure 9.05 shows how different values of the constant will affect the evolution of a flat, critically bound universe.

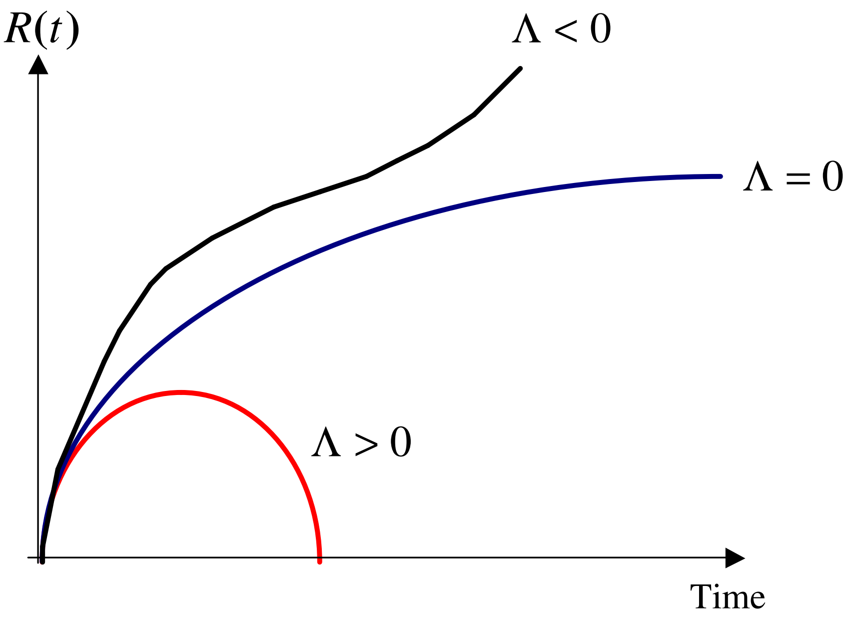


**Figure 9.05: Flat universes (*E* = 0) with different cosmological constants, Λ**

If we allow for the presence of the Cosmological Constant, our earlier statement that, in a flat universe $\Omega_m = 1$, must now be modified to the requirement:

$$\Omega_m + \Omega_\Lambda = 1 \,. \qquad (9.17)$$

It also modifies Equation 9.14 to:

$$q_0 = \frac{1}{2}\Omega_m - \Omega_\Lambda \,. \qquad (9.18)$$

### *9.3.6 Current estimates for cosmological parameters*

- Measuring the distance to, and recessional velocity of, distant galaxies or other objects allows for a determination of the Hubble constant $H_0$.

- The current best estimate[1] is $H_0 \approx 70$ km s$^{-1}$ mpc$^{-1}$.
  - This implies that, for a flat universe with $\Lambda = 0$, $t_0^{\text{flat}} = 9.3$ Gyr.
  - An empty universe (again with $\Lambda = 0$) will have $t_0^{\text{empty}} = H_0^{-1} = 14$ Gyr.

But what is $\Lambda$? The key to getting a handle on the values of $\Omega_m$ and $\Omega_\Lambda$ is to uncover an estimate of $q_0$.

Hubble's Law tells us that the recessional velocity is proportional to the distance to the observed object.

- If we observe very distant objects, we will be looking at a much earlier epoch when the rate of expansion will have differed from more recent epochs (if there is a non-zero acceleration or deceleration).

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[1] Mould J. R., et. al., 2000, ApJ, 529, 786

- If we can detect some deviation from the straight-line of Hubble's Law for very distant objects, we can hope to get a handle on the value of $q_0$, and in turn $\Omega_m$ and $\Omega_\Lambda$.

The Supernova Cosmology Project has observed > 40 distant supernovae.

Their data indicate that $\Lambda > 0$. They appear to be inconsistent with a flat, $\Lambda = 0$ model, and also do not fit well an open $\Lambda = 0$ cosmology.

If the results are constrained to a flat cosmology, that has $\Omega_m + \Omega_\Lambda = 1$, the best fit gives:

- $\Omega_m = 0.28$
- Since luminous matter accounts for about 5 per cent of the critical density, this result implies that dark matter makes up about $0.28 - 0.05\,/\,0.28 \approx 82$ per cent of all matter in the universe.
- $\Omega_\Lambda = 0.72$
- Therefore $q_0 = 0.28\,/\,2 - 0.72 = -0.58$
- The universe is accelerating!?