Dynamics of the Universe

The dynamical equations for a uniform, unbounded universe can be readily derived from a Newtonian treatment. Since this is so central to cosmology, a derivation of the key equations is provided below. This closely follows Liddle, who gives a good treatment.

Following the *Cosmological Principle*, we assume a uniform distribution of matter, and consider the expansion of a sphere of material of radius x and mass $M = \frac{4}{3}\pi x^3 \rho$.

In classical physics, *Newton's theorem* tells us that surrounding spherical shells of matter do not affect the dynamics inside the sphere. In General Relativity (which is needed to deal with distant shells, where the possible curvature of space cannot be ignored) a corresponding result known as *Birkhoff's theorem* has the same effect.

The total energy of a shell of mass m at the edge of the sphere is

$$E = V + T = -\frac{GMm}{x} + \frac{1}{2}m\dot{x}^2. \tag{1}$$

Since the whole Universe is expanding, it is helpful to separate the radius $x(t)$ of the sphere into a *comoving distance* r, and a *scale factor* $a(t)$ which describes the expansion of the Universe. So

$$x(t) = r\,a(t).$$

(The time coordinate t here, is the *cosmic time*, which can be agreed on by all comoving observers – e.g. it could be set by the evolving mean density of the Universe.)

We define r to be the real physical radius at the present epoch ($t = t_0$), so that a is dimensionless and $a(t_0) = 1$.

Warning: Different authors use different systems for defining the scale factor. Since the requirement is that $r\,a(t)$ has to be a real distance, it is possible to make either r or a dimensionless, and also to use different unit systems. Rowan-Robinson denotes the scale factor by R and gives it dimensions of length, but then divides it by its current value, R_0, so that our a is his R/R_0. Also, Rowan-Robinson uses a dimensionless k (whereas our k has units of length^{-2}), and Rowan-Robinson's (k/R_0^2) equals our k. Liddle's treatment is similar to ours, except that he denotes the comoving distance as x and the real distance as r. Most authors denote comoving distance by r (and in fact Liddle switches to this in his chapter A1), so we will follow this practice.

Substituting for x in (1) and dividing through by m, gives

$$\mathcal{E} \equiv E/m = -\frac{4\pi G\rho r^2 a^2}{3} + \frac{1}{2}r^2\dot{a}^2,$$

and rearranging,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} + \frac{2\mathcal{E}}{r^2 a^2}$$

which we write as

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H(t)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2}, \qquad \textit{the Friedmann equation.} \tag{2}$$

Here $k = -\frac{2\mathcal{E}}{r^2c^2}$, and $H(t)$ is the *Hubble parameter* (its value $H_0 \equiv H(t_0)$ at the present time is the *Hubble constant*). The last term in the Friedmann equation cannot depend on r (since the first two don't), hence k must be independent of r. Since $k = -\frac{2\mathcal{E}}{r^2c^2}$, it is also clearly independent of time (remember that $\mathcal{E}$ is conserved) – so k is actually a fundamental property of the Universe, and it turns out to be related to the curvature of space in GR.

To solve equation (2) for the evolution of the Universe, we need to know how ρ evolves with change in a. Due to mass-energy equivalence ($E = mc^2$), what we need is an energy equation for the expanding sphere of fluid. So, using

$$dE = T\,dS - P\,dV,$$

setting $E = mc^2 = \frac{4}{3}\pi a^3 r^3 \rho c^2$, and considering an adiabatic expansion ($dS = 0$), we have

$$\frac{dE}{dt} = -P\frac{dV}{dt},$$

so that

$$\frac{4}{3}\pi r^3 c^2 \frac{d}{dt}(a^3\rho) = -P\frac{4}{3}\pi r^3 \frac{da^3}{dt},$$

giving

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P/c^2) \tag{3}$$

Combining this with the Friedmann equation, we can derive a useful equation for the acceleration of the expansion. Differentiating (2)

$$\frac{2\dot{a}}{a}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) = \frac{8\pi G\dot{\rho}}{3} + \frac{2kc^2\dot{a}}{a^3}.$$

Substituting for $\dot{\rho}$ and dividing by $2\dot{a}/a$,

$$\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -4\pi G(\rho + P/c^2) + \frac{kc^2}{a^2},$$

and subtract off equation (2) to give

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P/c^2) \qquad \textit{the acceleration equation} \tag{4}$$

(**Warning:** Some authors adopt units such that $c = 1$ in this equation.)

Note from (4) that P does not help to *drive* expansion (there are no pressure *gradients* to do so) – it actually *retards* expansion (via work done). This point is important for any understanding of the effects of the cosmological constant.

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